Wiener Filters

The contents of this chapter are:

■ Introduction

In the **Digital Filters and Z Transforms** chapter we introduced inverse filters as a way of undoing some instrumental effect to determine the "true" signal. Later in that chapter, we saw that if we have a filter which has a large number of terms in it:

$$C[z] = c_0 + c_1 z + c_2 z^2 + ... + c_m z^m$$

$$Y[z] = C[z]\, X[z]$$

then we can do "another implementation" of the same filter in terms of its inverse that may have fewer significant terms in it:

$$\frac{1}{C[z]} = D[z] = \sum_{k=0}^{L_d} d_k z^k$$

$$y_i = \frac{x_i - \sum_{k=1}^{L_d} d_k y_{i-k}}{d_0}$$

In this chapter we return to the original motivation for discussing these filters, in particular the problem of removing noise from some signal.

For generality, we assume that there are two processes that effect the signal that we measure m_k.

First, assume that the apparatus is not perfect but instead smears out the signal somewhat. If the "true" signal is t_k and the response function of the apparatus is r_k then we get a smeared out signal s_k:

$$s_k = r_k * t_k \Leftrightarrow S_j = R_j T_j$$

where as usual we write * for convolution and display the result in both the time domain and the frequency domain. Also, as usual, we assume the sampling interval Δ is one; otherwise sums in the time domain below need to be multiplied by Δ and sums in the frequency domain need to be divided by Δ.

Second, assume that the smeared out signal has had noise n_k added to it, giving us the measured signal m_k:

$$m_k = s_k + n_k$$

In the absence of noise, if we know the response function of the apparatus we already know how to find the true signal:

$$T_j = \frac{S_j}{R_j}$$

Now we want to find the *optimal Wiener filter*, w_k or W_j which, when applied to the measured signal and deconvolved by the instrument response, gives us an estimate of the true signal:

$$\tilde{T}_j = \frac{W_j\, M_j}{R_j}$$

Note that W_j is an inverse filter: it gives us the smeared signal from the apparatus from the measured signal.

■ Least-Squares Regression

Above we set up the problem we wish to solve, finding an estimate $\tilde{t}_k$ of a true signal t_k. Here we examine in what sense our estimate is close to the true signal.

The *residual* is the difference between the true signal and the estimated one:

$$\text{Residual}_k = t_k - \tilde{t}_k$$

and the *sum of the squares* of the residuals is:

$$\text{SumOfSquares} = \sum_{k=0}^{n-1} \left(t_k - \tilde{t}_k\right)^2$$

Least-squares regression, used for example by most curve fitters, minimises the **SumOfSquares** by taking a series of derivatives of it with respect to the parameters we are trying to determine and setting each equal to zero:

$$\partial_{\tilde{t}_0} \text{SumOfSquares} = 0$$

$$\partial_{\tilde{t}_1} \text{SumOfSquares} = 0$$

$$\partial_{\tilde{t}_2} \text{SumOfSquares} = 0$$

...

$$\partial_{\tilde{t}_{n-1}} \text{SumOfSquares} = 0$$

For the case of fitting data $\{\{x_1, y_1\}, \{x_1, y_1\}, \ldots, \{x_1, y_1\}\}$ to, say, a straight line:

$$y = m\,x + b$$

the residuals are:

$$\text{Residual}_k = y_k - (m\,x_k + b)$$

and the minimum of the sum of the squares of the residuals is the solution to the equations:

$$\partial_b \text{SumOfSquares} = 0$$

$$\partial_m \text{SumOfSquares} = 0$$

For this example, the solution of the equations is analytic.

For the Wiener filter, we similarly want our estimate of the true signal $\tilde{t}_k$ to be close to the actual value t_k in the same sense that the sum of the squares of the residuals is a minimum.

$$\sum_{k=0}^{n-1} \left(t_k - \tilde{t}_k\right)^2 = \sum_{j=0}^{n-1} (T_j - \tilde{T}_j)^2 \text{ is minimised}$$

There is a potentially important caveat in this and all least-squares techniques. This is that they are not *robust*, by which we mean that a single "wild" data point can seize control of the whole analysis, giving non-sensical results.

This is similar to the fact that the mean or average is not robust, although the median is. This similarity is not coincidental. Say we repeat a measurement of, say, the diameter of a metal hoop **n** times:

$$\{x_1, x_2, x_3, \ldots, x_n\}$$

Call the estimated value $\overline{x}$. Then the sum of the squares is:

$$\text{SumOfSquares} = \sum_{k=1}^{n} (x_k - \bar{x})^2$$

The minimum is when:

$$\partial_{\bar{x}}\ \text{SumOfSquares} = 2 \sum_{k=1}^{n} (x_k - \bar{x}) = 0$$

or:

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + ... + (x_n - \bar{x}) = 0$$

Therefore:

$$\bar{x} = \frac{(x_1 + x_2 + x_3 + ... x_n)}{n}$$

But this is exactly the mean. Therefore, the mean is a least-squares estimate of the expectation value for repeated measurements.

■ Implementation

We have just seen that we wish to find a Wiener filter W_j that minimises:

$$\sum_{j=0}^{n-1} (T_j - \tilde{T}_j)^2$$

where:

$$T_j = \frac{S_j}{R_j}$$

and:

$$\tilde{T}_j = \frac{W_j\, M_j}{R_j}$$

But the measured signal M_j is just the smeared signal S_j plus the noise N_j so we are trying to minimise:

$$\sum_{j=0}^{n-1} \left(\frac{S_j}{R_j} - \frac{W_j\,(S_j + N_j)}{R_j} \right)^2 = \sum_{j=0}^{n-1} \frac{1}{R_j^2} \left(S_j^2\,(1 - W_j)^2 + N_j^2\, W_j^2\right)$$

Now, this has a term involving the smeared signal and a second term involving the noise, and includes all the frequencies ω_j. However the smeared signal and the noise are uncorrelated: this is practically the definition of noise. Thus if the above equation is to be a minimum it must be a minimum for each and every term in the sum.

We have *Mathematica* take the derivative of a term with respect to the Wiener filter and then solve for when that derivative is equal to zero:

$$\textbf{derivative} = \partial_{W_j}\left(\frac{1}{R_j^2}\left(S_j^2\,(1 - W_j)^2 + N_j^2\,W_j^2\right)\right)$$

$$\frac{-2\,S_j^2\,(1 - W_j) + 2\,N_j^2\,W_j}{R_j^2}$$

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Solve[derivative == 0, Wj]
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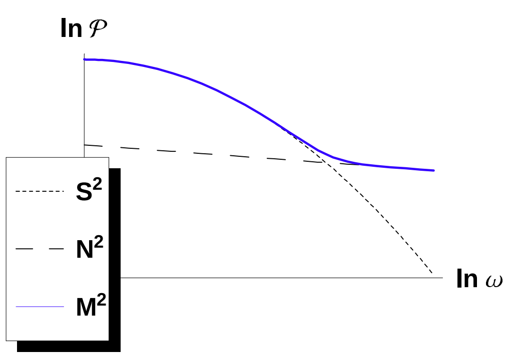
$$\left\{\left\{W_j \to \frac{S_j^2}{N_j^2 + S_j^2}\right\}\right\}$$

This is the optimal Wiener filter. Note that the answer does not contain the "true" signal T_j. Thus the filter can be determined independently of the response function of the apparatus R_j.

In order to determine the Wiener filter, we realise that since the smeared signal and the noise are independent, the power $\mathcal{P}$ is proportional to the sum $S^2 + N^2$, i.e. the cross terms **2SN** are zero. If we measure a time series such that the Nyquist frequency is much greater than the maximum frequency of the smeared signal, then a log-log plot will look like the continuous line in the following figure:



The point is that by looking at the measured signal at large frequencies, we can extrapolate back to make a reasonable eyeball estimate of the noise at all frequencies. Similarly, for small frequencies we can similarly estimate the smeared signal at larger frequencies; this estimate should match the measured signal squared minus the noise signal squared.

■ Conclusion

The Wiener filter implementation is not as sophisticated as the methods we discussed in the previous chapter. However there are some things in its favor.

First, it uses the great power of our visual system as a natural integrator and extrapolator to determine the noise signal. Despite the claims sometimes seen in glossy advertisements, our visual system is still able to beat computers in pattern recognition tasks.

Second, it turns out that the Wiener filter differs from the true optimal filter by an amount that is second order in the precision to which the filter is determined. This means that even a fairly sloppy determination can still give excellent results.

In fact, Wiener filters are one of the tools of choice in removing noise from photographic images.

■ Author