Mass-Energy Equivalence and Relativistic Inelastic Collisions

Jason Harlow and David M. Harrison
Department of Physics
University of Toronto

Introduction

Einstein was led to mass-energy equivalence by considering the interaction between a charged particle and an electromagnetic field.[1] His original argument is fairly complex for beginning students. A few years ago one of us (DMH) devised a simpler way of demonstrating that the mass of a body must increase with its speed. The description considers a totally inelastic collision between two equal masses and only requires knowing about conservation of momentum and that relativistic speeds do not add in a simple way. It has proved to be effective with Physics students; it also works well in liberal arts courses that use little or no mathematics.

However, when we tried to make this description quantitative we were surprised to discover that a simple-minded approach did not work. It turns out that to conserve momentum in all frames it must be assumed that the kinetic energy lost during the collision is converted to mass.

We will first describe the qualitative argument. We then go through the quantitative analysis and will associate the dissipated energy with the mass of the system.

The Qualitative Argument

Figure 1 shows two objects with equal rest masses undergoing a perfectly inelastic collision as viewed from a reference frame in which the total momentum is zero. We will call this Frame 1. Before the collision the speed of each object is 0.6 c, and after the collision the two objects are stationary. All the initial kinetic energy has been converted to energy of heat and deformation.

Figure 2 shows the same collision in a frame of reference moving to the left at 0.6 c relative to the first frame. We call this Frame 2. In this frame, before the collision object 1 is moving at a speed given by the relativistic formula for addition of velocities:

$$v' = \frac{2v}{1 + vu/c^2} \tag{1}$$

In this case, $v = u = 0.6c$, *and* $v' = 0.88c$.

Before the collision object 2 is stationary in Frame 2. Since after the collision the two objects are stationary in Frame 1, the principle of relativity requires that they are moving to the right at 0.6 c in Frame 2.

If before the collision the mass of 1 did not increase as seen in Frame 2, then conservation of momentum would lead to the conclusion that the speed of the two masses after the collision must be one-half of v', 0.44c. Thus, to preserve the principle of conservation of momentum we must conclude that the mass of 1 before the collision is greater than its rest mass.

Quantitative Analysis

In Frame 1, the total energy before the collision is:

$$E_i = 2\frac{m_0}{\sqrt{1-v^2/c^2}}c^2 = 2.5m_0c^2 \tag{2}$$

After the collision one might be tempted to write that the total energy is 2 $m_0 c^2$, which is less than the initial energy. The energy difference, $\Delta E = 0.5\, m_0\, c^2$, has gone into energy of deformation and heat, perhaps being radiated away from the masses. We will find it useful to associate this energy with the masses themselves, and form an effective mass:

$$m_{eff} = 1.25m_0 \tag{3}$$

Then the total final energy is equal to the initial energy:

$$E_f = 2m_{eff}c^2 = 2.5\, m_0c^2 \tag{4}$$

In Frame 2, the initial total momentum is just the momentum of the object 1:

$$p_i = \frac{m_0}{\sqrt{1-v'^2/c^2}}v' = 1.875\, m_0c \tag{5}$$

The final total momentum of the objects, now stuck together, is:

$$p_f = 2\frac{m}{\sqrt{1-v^2/c^2}}v = 1.50\, mc \tag{6}$$

In Eqn. 6, if we set the mass *m* to be equal to the rest mass there is an apparent violation of conservation of momentum. It was this that surprised us, and led us to introduce the effective mass. We will find its value from conservation of energy evaluated in Frame 2:

$$E_i = \left(\frac{m_0}{\sqrt{1 - v'^2 / c^2}} + m_0\right)c^2 = 3.125\, m_0 c^2 \tag{7}$$

$$E_f = 2\frac{m_{eff}}{\sqrt{1 - v^2 / c^2}} c^2 = 2.50\, m_{eff} c^2 \tag{8}$$

Setting the initial energy equal to the final energy and solving for the effective mass gives:

$$m_{eff} = 1.25 m_0 \tag{9}$$

Thus momentum is conserved in Frame 2:

$$p_f = 1.50 \times 1.25 m_0 c = p_i \tag{10}$$

It is interesting to note that the effective mass is invariant, and has the same value in all reference frames.

It is possible to think of the difference between the effective mass and the rest mass as the heat energy of the object in the "rest" Frame 1 divided by c^2. If the speed of the two objects in Frame 1 is v, setting the algebraic forms of Eqns. 2 and 4 equal to each other and solving for $m_{eff} - m_0$ gives:

$$m_{eff} - m_0 = \left(\frac{1}{\sqrt{1 - v^2 / c^2}} - 1\right) m_0 \tag{11}$$

This result can also be derived by using conservation of energy in Frame 2, Eqns. 7 and 8, and using the relativistic addition of velocities, Eqn. 1, to eliminate v'.

With beginning students, taking a relativistic relation and showing that for small speeds the result is the non-relativistic answer is usually worthwhile. Expanding Eqn. 11 and throwing out higher order terms gives the non-relativistic result:

$$m_{eff} - m_0 \cong \frac{1}{2} m_0 \frac{v^2}{c^2} \tag{12}$$

Thus the total energy H lost in the collision is:

$$H = 2(m_{eff} - m_0)c^2 \cong 2 \times \frac{1}{2} m_0 v^2 \tag{13}$$

This is correct: in Frame 1 all the kinetic energy is converted to heat.

We should emphasise that we have only described the invariant heat, the value in the center of mass Frame 1. The Lorentz transformations for heat are complex, and beyond the level of this discussion.[3]

Conclusions

As already mentioned, the qualitative part of the above is classroom tested. The course notes in Ref. 2 are a more student-friendly version of the discussion, with a large extension involving the dimensionality of spacetime and the momentum vector.

In retrospect, the quantitative analysis should perhaps not have been much of a surprise to us, but it was. The problem totally stumped some of our colleagues. It also surprises students, either in class or a problem set. The resolution of the surprise reinforces the whole notion of mass-energy equivalence, in this case by associating a mass with dissipated energy. The fact that at least some of this energy can be a radiating electromagnetic field is still somewhat of a surprise to us.

We have not derived the algebraic relationship between mass, speed, and rest mass in our analysis, but only used the result. Feynman does an elegant derivation.[4]

If the collision does not occur in a vacuum, there may also be energy radiated away as a sound wave. Of course, in this case there will also be energy dissipation due to the air resistance, which would need to be accounted for in the analysis.

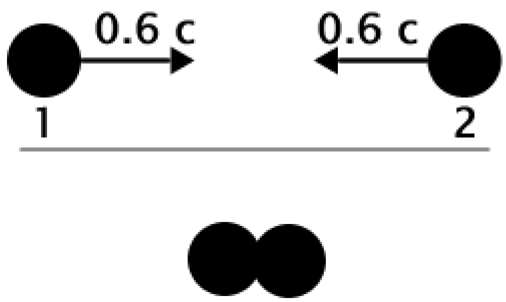
References

1. A. Einstein, Annalen der Physik **17**, 1905. Translated and reprinted as “Does the inertia of a body depend on its energy content?” in H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl**, The Principle of Relativity**, W. Perrett and G.B. Jeffery trans., (Dover, New York), p. 66. There are many copies of this paper on the web.
2. Course notes from an upper-year non-mathematical liberal arts course are at: http://www.upscale.utoronto.ca/PVB/Harrison/SpecRel/MassEnergy.html.
3. A recent review is B. Lavenda, “Does the inertia of a body depend on its heat content?**”** Naturwissenschaften **89**, 329 (2002).
4. R. P. Feynman, R.B. Leighton and M. Sands, **The Feynman Lectures on Physics**, Vol. I (Addison-Wesley, Toronto), §16-4.

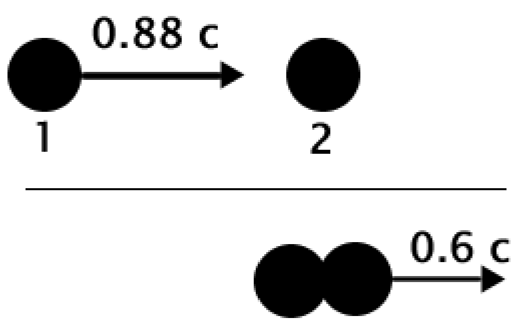
Figure Captions

Figure 1. Two equal masses undergoing a totally inelastic collision as seen in the center of mass reference frame. The upper part is before the collision, and the lower part is after.

Figure 2. The same collision as in Figure 1 as seen in a reference frame where the right hand object 2 is initially stationary.



(Figure 1)



(Figure 2)