PHY 353 S PROBLEM SET 2

1 Starting with $c = \nu\lambda = 2\pi\nu \cdot \frac{\lambda}{2\pi} = \frac{1}{2\pi}\omega\lambda$

then $\omega = \frac{2\pi}{\lambda} c$

So $d\omega = 2\pi c \cdot (-1) \frac{1}{\lambda^2} d\lambda$ (assuming $\lambda >> d\lambda$ allows us to compute this differential)

Now the resolving power is $R = \frac{\lambda}{\delta\lambda} \simeq \lambda\left(-\lambda^2 d\omega/2\pi c\right)^{-1}$

$= \frac{-1}{\lambda d\omega} \cdot 2\pi c = -\frac{\omega}{d\omega}$. (neg. sign is irrelevant, it only depends on definition of $d\omega$).

2. Let x_m be the radius of the mth Newton's ring.

Thus,

$$x_m = \left[(m+\tfrac{1}{2})\lambda R\right]^{\frac{1}{2}} \quad (*)$$

We are given that $m = 20$, $x_m = 1\text{cm}$, $\lambda = 500\text{nm}$

So, we can solve for R

$$x_m^2 = (m+\tfrac{1}{2})\lambda R$$

$$R = \frac{x_m^2}{(m+\frac{1}{2})\lambda} = 9.75\text{ m}$$

This demonstrates the power of using N's rings to determine flatness → such a large curvature would be difficult, if not impossible to discern using the eye.

3 In the derivation of the single layer AR coating, it was found that for normal incidence, the thickness of the coating is optimally $\frac{\lambda}{4n_1}$ (i.e. Hecht p.420, and you may have also derived this result in PHY 255 F). n_1, of course, is the refractive index of the coating.

Using $\lambda = 500\text{nm}$, we find the AR film thickness should be $\frac{125\text{nm}}{n_1} = \frac{125\text{nm}}{1.33} = 94\text{nm}$.

The media were not clearly defined in the question, so it's unclear if $\lambda = \lambda_{air}, \lambda_{glass}, \lambda_{film}$, so I basically accepted anything.

#4. As the angle of incidence approaches $\frac{\pi}{2}$, $R \to 1$, as evidenced directly from the Fresnel equations. However, that's not the end of the story, since for $\theta_i > \theta_c$, light will be completely reflected.

Generally $\theta_c < \frac{\pi}{2}$

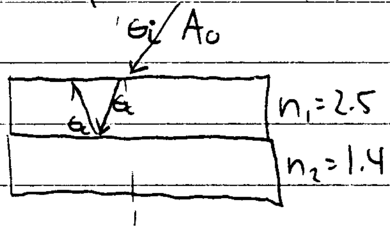
But, in going from air to the first slide ($n_1 = 2.5$)

$$\sin\theta_c = \frac{2.5}{1} \quad \text{and no real } \theta_c \text{ exists!}$$

However, in going from $n_1 = 2.5$ to $n_2 = 1.4$

$$\sin\theta_c = \frac{1.4}{2.5} \qquad \theta_c = 34^\circ$$

Thus, we should find θ_i, s.t. $\theta_c = 34^\circ$



$$\sin\theta_i = n_1 \sin\theta_c$$

$$\sin\theta_i = 1.4$$

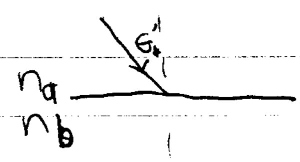
still no real θ_c!

Thus, we can't rely on TIR to get the job done.

Finally, as $\theta_c \to \frac{\pi}{2}$, $R \to 1$, however, there ~~must always~~ may be an evanescent wave in order to satisfy the boundary conditions. The amplitude of the evanescent wave varies as

$$A(z) = A_0 \exp(-\beta z)$$

(*) $\beta = \frac{\omega}{c}(n_a^2 \sin\theta' - n_b^2)$



as $\sin\theta'_i \to 1$, $\beta \to \frac{\omega}{c}\sqrt{n_b^2 - n_a^2}$

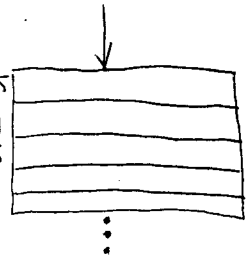
and in going from air to $n_1 = 2.5$

$$\beta = \frac{2\pi\nu}{c}\sqrt{4.29} = \frac{2\pi \cdot 5 \cdot 10^{14}}{c}\sqrt{4.29}$$

$$= 2.2 \cdot 10^{+7}\ m^{-1}$$

(*) note!! β is imaginary! β has to be real! This means there is NO evanscent wave. Since no θ_c existed, this is not surprising. The rest of the solution is a fib.

4. Expressions for reflectivity are generally complicated, but assuming the microscope slides are lossless, and assuming normal incidence.



$$R = |r|^2 = \left(\frac{n_1-n_2}{n_1+n_2}\right)^2$$ in which the light passes from medium with ref. index n_1 to that with ref. index n_2

However, the expression above is for one boundary only. We have ten here.

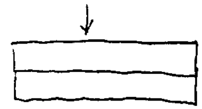
Thus, consider the following more general expression (usually used for thin film)

$$r = \frac{Y_0 m_{11} + Y_0 Y_s m_{12} - m_{21} - Y_s m_{22}}{Y_0 m_{11} + Y_0 Y_s m_{12} + m_{21} + Y_s m_{22}}$$

Y_0 = admittance of air
Y_s = admittance of substrate $= Y_0$ (which we'll take to be air)

$$\bar{\bar{M}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$ = transfer matrix of system (the 10 slides)

The transfer matrix for one high-index/low-index system is $\bar{\bar{M}}_1 = \begin{bmatrix} -n_1/n_2 & 0 \\ 0 & -n_2/n_1 \end{bmatrix}$



For five of these (our case)

$$\bar{\bar{M}}_5 = \begin{bmatrix} -\left(\frac{n_1}{n_2}\right)^5 & 0 \\ 0 & -\left(\frac{n_2}{n_1}\right)^5 \end{bmatrix}$$ (simple to calculate since $\bar{\bar{M}}_1$ is diagonal)

sub these matrix elements into the expression for r, with $Y_0 = Y_s$

$$r = \frac{-Y_0\left(\frac{n_1}{n_2}\right)^5 + Y_0\left(\frac{n_2}{n_1}\right)^5}{-Y_0\left(\frac{n_1}{n_2}\right)^5 - Y_0\left(\frac{n_2}{n_1}\right)^5} \quad \frac{\left(\frac{n_1}{n_2}\right)^5 - \left(\frac{n_2}{n_1}\right)^5}{\left(\frac{n_1}{n_2}\right)^5 + \left(\frac{n_2}{n_1}\right)^5} = \frac{n_1^{10} - n_2^{10}}{n_1^{10} + n_2^{10}}$$
$$= 0.9939$$

$\therefore$ min reflectance for TE polarization is $|r|^2 = 0.9879$. We know that reflectivities for TE waves increase monotonically as θ_i goes from $0 \to \frac{\pi}{2}$ for TM waves, there will be a minimum at the Brewster angle ($R=0$) So, for arbitrarily polarized light, this answer above may not be the minimum reflectance (it will occur near the θ_B)

I'm now going to blatantly ignore the fact that the slides are alternating high and low refractive indices. Sure, the wavevector will change by going from one medium to the next, but frankly, who really feels like splitting hairs on this?

Estimate each slide to have a thickness of 1mm

Thus, $A(z=10\,mm) = A_0 \exp\left((-2.2\times10^{+7} m^{-1})(1\times10^{-2} m)\right)$

$$= A_0 \exp(-2.2\times10^{+5})$$

$$\doteq 0$$

So, the transmitted field amplitude coefficient is $\exp(-2.2\times10^{5}$

$\therefore$ The peak reflectance is $R = 1-\exp(-4.4\times10^{+5})$

However, as I said before, this is a fib, because there seems to be NO evanescent wave.

Thus, the peak reflectance is simply from the Fresnel equations, which dictates $R\to1$ as $\theta_i \to \frac{\pi}{2}$.

5. The power spectrum of an electric field is the Fourier transform of the radiated electric field (Parseval's theorem): Implied

So $F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt$

$$= \int_{-\infty}^{\infty} A\, e^{-at^2 - i\omega_0 t}\, e^{i\omega t}\, dt$$
$$= A \int_{-\infty}^{\infty} e^{it(\omega-\omega_0)}\, e^{-at^2}$$

Turning to a table of integrals (you didn't think I was going to evaluate this by hand, did you?) - Schaum's Mathematical Handbook p.98 #15.75

$$\int_{-\infty}^{\infty} e^{-at^2}\, e^{-bt}\, e^{-c}\, dt = \sqrt{\frac{\pi}{a}}\, e^{b^2-4ac/4a}$$

Making the substitutions $a \leftrightarrow a$ (sorry for the confusion in notatio

$b \leftrightarrow i(\omega_0 - \omega)$

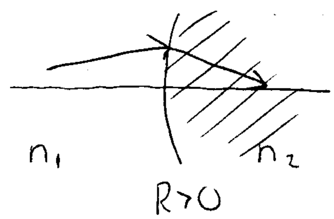
$c \leftrightarrow 0$

We get

$$F(\omega) = A\sqrt{\frac{\pi}{a}}\, \exp\left(\frac{(i(\omega_0-\omega_0))^2}{4a}\right)$$
$$= A\sqrt{\frac{\pi}{a}}\, \exp\left(\frac{-(\omega_0-\omega)^2}{4a}\right)$$

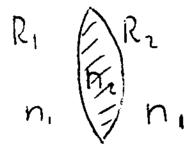
which is a Gaussian pulse centred on ω_0.

6. The ray matrix for a curved dielectric interface is



$$\begin{bmatrix} 1 & 0 \\ \frac{n_1-n_2}{n_2R} & \frac{n_1}{n_2} \end{bmatrix}$$

The system we are considering is



Note if we define $R_1>0$, then $R_2<0$

At the first interface we go from air $(n_1=1)$ to glass $(n_2=n)$. The ray transfer matrix is

$$\begin{bmatrix} 1 & 0 \\ \frac{1-n}{nR_1} & \frac{1}{n} \end{bmatrix}$$

Then from glass to air through the second interface *

$$\begin{bmatrix} 1 & 0 \\ \frac{n-1}{(+R_2)} & n \end{bmatrix}$$

So, the total ray transfer matrix is

$$\overline{\overline{M}}_{tot} = \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1-n}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{(1-n)}{-R_2} + \frac{1-n}{R_1} & 1 \end{bmatrix}$$

$$\overline{\overline{M}}_{tot} = \begin{bmatrix} 1 & 0 \\ (1-n)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) & 1 \end{bmatrix}$$

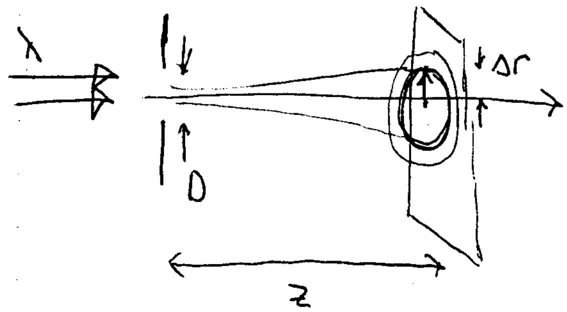
A lens of focal length f has the transfer matrix

$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$, so by letting $f = \left((n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right)^{-1}$ in $\overline{\overline{M}}_{tot}$, we see that $\overline{\overline{M}}_{tot}$ represents a lens of focal length f, given by the expression above.

* since the lens is "thin", we will neglect the ray propogation in the glass.

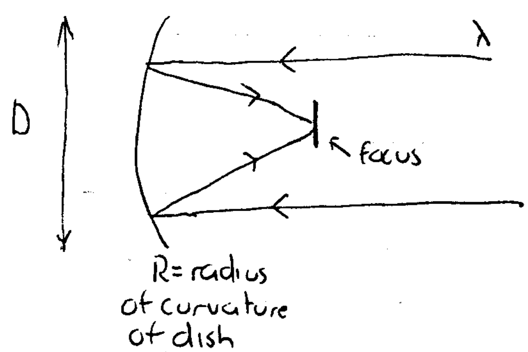
7. Consider light incoming from an infinitely distant source. After passing through an aperture of diameter D, the width of the Airy disk is

$$\Delta r = \frac{1.22 \lambda z}{D} \quad (*)$$



If the light is collected by a lens, then $z \approx f$ (since light is incoming from ∞), $\Delta r = \frac{1.22 \lambda f}{D}$

In a telescope, we have

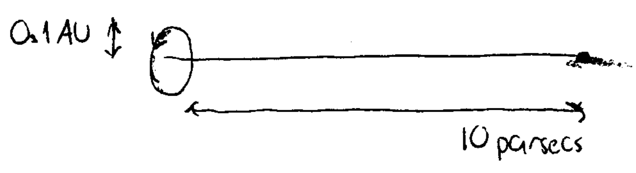


Applying Babinet's principle, we see that the situation at left is complimentary to the case of the aperture, above

So, we can apply (*), above to the telescope.
The angular deviation, $\frac{\Delta r}{z} = \Delta\theta$ can be subbed in

$$\Delta\theta = \frac{1.22 \lambda}{D}$$, as required.

This is what we need to resolve: (not to scale)



1 parsec $= 3.1 \times 10^{16}$ m
1 A.U $= 1.5 \times 10^{11}$ m

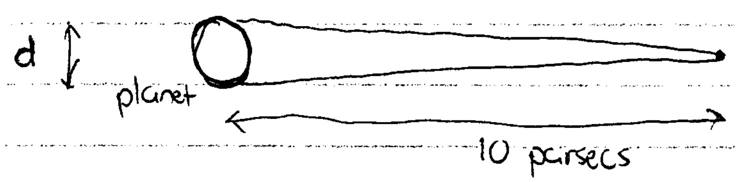
$$\text{so, } \Delta\theta \simeq \frac{0.1 \times 1.5 \times 10^{11}\text{ m}}{10 \times 3.1 \times 10^{6}\text{ m}} = \frac{1.22\lambda}{D}$$

Let's take $\lambda = 5.5 \times 10^{-7}$ m (centre of visible spectrum)
Solving for D,

$$D = \frac{1.22(5.5 \times 10^{-7}\text{ m})(10)(3.1 \times 10^{+16}\text{ m})}{0.1 \times 1.5 \times 10^{11}\text{ m}}$$

$$= 13.9\text{ m}.$$

To produce a crude image of the planet, we must resolve:



In this case, if we estimate the diameter of the planet to be 1×10^{7} m (of the order of the earth's diameter),

$$\Delta\theta = \frac{1 \times 10^{7}\text{ m}}{10 \times 3.1 \times 10^{16}\text{ m}} = \frac{1.22\lambda}{D}$$

$$D = 2.1 \times 10^{4}\text{ m} \simeq 21\text{ km}!$$

To circumvent this problem, people build many small telescopes and line them up. The net effect is the resolving power of a larger telescope. Of course, there is a phase difference between light rays hitting each telescope, but this can be accounted for if the telescopes' positions are known (obviously, this is true). In principle, it would also be possible to build a "telescope array".

8. The Doppler shift due to reflection off of the planet is $1\times10^5 \frac{m}{s}$
So, the frequency of the reflected light is shifted:

$$\nu' = \nu_0 (1 \pm \tfrac{v}{c})$$ where ν' is the shifted freq.
ν_0 is the directly emitted freq.

$$\Delta\nu = \nu' - \nu_0 = \nu_0(1 \pm \tfrac{v}{c}) - \nu_0 = \pm\nu_0 \frac{v}{c}$$

To convert into wavelength, remember $\nu = \frac{c}{\lambda}$

$$\Delta\nu = -c\frac{1}{\lambda^2}\Delta\lambda$$

Making the substitutions,

$$-c\frac{1}{\lambda_0^2}\Delta\lambda = \pm\nu_0\frac{v}{c} = \pm\frac{c}{\lambda_0}\frac{v}{c}$$

$$\Delta\lambda = \mp\lambda_0\frac{v}{c}$$

Given that $\lambda_0 = \cancel{8}6439.07\,\text{Å}$, $v = 1\times10^5 \frac{m}{s}$
then

$$\Delta\lambda = \mp 2.15\,\text{Å}$$

So, we need an FTI with a $2.15\,\text{Å}$ spectral resolution.

We can estimate $\Delta\omega\Delta t \sim 1$
And now we calculate the required <u>time</u> resolution

$$\Delta\nu\Delta t \sim 2\pi$$

$$\frac{c}{\lambda^2}\Delta\lambda\Delta t \sim 2\pi$$

$$\Delta t \sim \frac{2\pi\lambda^2}{c\Delta\lambda}$$

this corresponds to a path length difference

$$\Delta s = c\Delta t = \frac{2\pi\lambda^2}{\Delta\lambda}$$

$$= 1.2\text{ cm}$$

(This is not exactly what some of you got, but it depends on the initial estimate, i.e. $\Delta\omega\Delta t \sim \frac{1}{2}$ or $\sim \frac{1}{2\pi}$, etc.)

from question 1

9. We require the resolving power $R.P. = \frac{\omega}{\Delta\omega} \simeq \frac{\lambda}{\Delta\lambda} > \frac{6439.07}{2.15} = 2995$

$$R.P. = \frac{\omega F n d \cos\theta}{1.05\pi c}$$

The instrument is evacuated (take $n=1$)
Take the incident angle, θ, to be $0°$
Substitute $\omega = 2\pi\nu = 2\pi\frac{c}{\lambda}$ and we have

$$R.P. = \frac{2Fd}{1.05\lambda} > 2995 \simeq 3\times10^3$$

We need $F > 3\times10^3 \cdot \frac{1.05\lambda}{2d}$

If the largest possible FP we can build is $d = 0.5m$, then
$F > 2\times10^{-3}$ Man, that's REALLY SMALL!
This means $\frac{\pi\sqrt{R}}{1-R} > 2\times10^{-3}$ $R < 0.18$!
The $FSR = \omega' - \omega_0 = \frac{\pi c}{nd\cos\theta} \rightarrow \frac{\pi c}{d}$

thus $\Delta\omega = FSR = 2\pi\Delta\nu = 2\pi\frac{c}{\lambda^2}\Delta\lambda = \frac{\pi c}{d}$ etc.

Simply $FSR = \frac{\pi c}{d} = 1.88\times10^9 s^{-1}\omega$

around $\lambda = 6.43\times10^{-7}m$, this corresponds to
a $\Delta\lambda = \lambda^2 \cdot \frac{1}{2d} = 4.13\times10^{-13}m$

What this means is that an FPI this size is overkill — it is more than capable of resolving the ~2Å difference in the direct and reflected spectra (even if the mirrors are really crappy)

10 This is essentially a units question.

$|I| = cn\epsilon_0 \langle E^2(t) \rangle$ using MKS units $= \frac{cn}{2}\epsilon_0 E^2$ (after taking the time average)

$|I| = 2.66 \times 10^{-3} E^2$ if $n=1$

$$E = 19.4\sqrt{|I|} \ \frac{V}{m}$$

We are given $I = \frac{1W}{(10^{-3}m)^2} = 10^6 \frac{W}{m^2}$

So $E = 19400 \frac{V}{m}$

and since

$E = cB \rightarrow B = \frac{E}{c} = 6.43 \times 10^{-5}$ T

but $H = \frac{B}{\mu_0} \rightarrow H = 51.68 \frac{A}{m}$

i.e. if I work in my lab and I have my wallet in my pocket, then I don't have to worry about my credit cards.

11 The mathematics involved is in Hecht and about a million other textbooks. Here's what you should understand: Huygen's Principle is an EXACT* formulation of diffraction (albeit qualitative)*. The Kirchhoff Integral is just Huygen's principle stated using fancier mathematics.

Unfortunately, the KI is unfriendly to any practical calculation. So, in deriving the Fresnel-Kirchhoff diffraction formula, we assume

a) waves are spherical waves emanating from a point source

b) the point source is "far" from the aperture(s) of interest, and the aperture(s) is/are "far" from the observation point

"Far" means that the scale of the lengths of interest (described in the previous sentence) is much larger than the scale of the wavelength.

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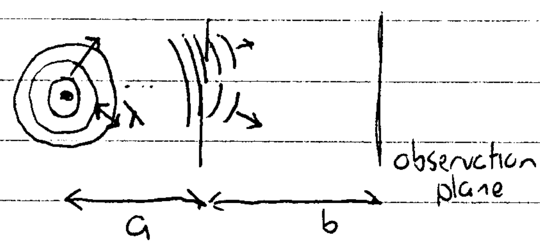
* it also assumes the Electric field to be scalar, so it's not QUITE exact. More specifically, Huygen's principle and Kirchhoff's Integral

(continued)

* continued assume that the apertures are non-absorbing, their polarizabilit[y] is zero, or alternately, the field is very weak. Sommerfeld, I believe, was the first to calculate diffraction integrals without using these assumptions. Let's just say those were the days before TV and the internet. Thus, the KI is exact enough for our purposes.

That was kind of garbled, so here's a diagram:



Essentially, $a, b \gg \lambda$

For visible light, the Fresnel approx. is valid for $a, b \gtrsim 1\text{cm}$. The Fraunhofer approx. is a stronger approx $\rightarrow$ it assumes that we have plane (not spherical) waves at the aperture.

12 Another units question.

Radiance has units of $\frac{W}{m^2 sr}$ (sr = steradian)

Irradiance has units of $\frac{W}{m^2}$ (usually called intensity)

So, if we integrate L over a hemisphere:

def $I = \int L \cos\theta \, d\Omega$ ← hemisphere $\therefore \theta \in \{0, \frac{\pi}{2}\}, \phi \in \{0, 2\pi\}$

$$= \int_0^{\pi/2} L \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$I = L \int_0^{\pi/2} \tfrac{1}{2} \sin 2\theta \, d\theta \int_0^{2\pi} d\phi = L \cdot \tfrac{1}{2} \cdot 2\pi = \pi L$$