PHY 353S PROBLEM SET 1 ANSWERS

1 We can write a monochromatic plane wave in the following form:

$$\vec{E} = E_0 \hat{a} \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$$

note that $\hat{a}$ denotes a unit vector

Thus,

i) the direction of the oscillating $\vec{E}$ field → given by $\hat{a}$

In the expression given, we have $-3\hat{i} + 3\sqrt{3}\hat{j}$ but this is not normalized

Normalizing → $\hat{a} = \frac{1}{\sqrt{3^2+(3\sqrt{3})^2}}(-3\hat{i} + 3\sqrt{3}\hat{j}) = \frac{1}{6}(-3\hat{i} + 3\sqrt{3}\hat{j})$

$$= -\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

(ii) given by E_0 but we must account for the factor of 6 which came from normalizing $\hat{a}$.

$$\therefore E_0 = 6\times10^4$$

(iii) the direction of propogation is $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$

$$\vec{k}\cdot\vec{r} = \frac{1}{3}(\sqrt{5}x + 2y)\pi\times10^7$$

let $\vec{k} = a\hat{i} + b\hat{j}$, a,b constants

since $\vec{r} = x\hat{i} + y\hat{j}$,

$$a = \frac{\sqrt{5}}{3}\pi\times10^7 \quad , b = \frac{2}{3}\pi\times10^7$$

the magnitude of $\vec{k}$ is

$$|\vec{k}| = \sqrt{(10^7\pi)^2\left(\left(\frac{\sqrt{5}}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right)}$$

$$= 10^7\pi$$

Thus, the direction of propogation $\hat{k} = \frac{1}{10^7\pi}\left(\frac{\sqrt{5}}{3}\pi\times10^7\hat{i} + \frac{2}{3}\pi\times10^7\hat{j}\right)$

$$= \frac{\sqrt{5}}{3}\hat{i} + \frac{2}{3}\hat{j}$$

iv) the propogation wavevector is just the unnormalized $\vec{k}$ (above) but since $k = \frac{2\pi}{\lambda}$ $(k = |\vec{k}|)$

then $\lambda = \frac{2\pi}{k} = \frac{2\pi}{10^7\pi} m = 2\times10^{-7} m = 200 nm.$

v) ω is angular frequency

from the given expression, $\omega = (9.42\times10^{15})\ s^{-1}$

s^{-1}

since frequency $f = \frac{\omega}{2\pi}$, then $f = 1.5\times10^{15}\ s^{-1}$

vi) the speed is $v = \frac{\omega}{k}$, so using the numbers from (iii) and (v)

$$v = \frac{9.42\times10^{15}\ s^{-1}}{10^7\pi\ m^{-1}} = 3\times10^8 \frac{m}{s}\ (=c)$$

2 (i) $I = \langle \vec{S} \rangle = 1.37\ kW/m^2$

the time averaged pressure is defined as

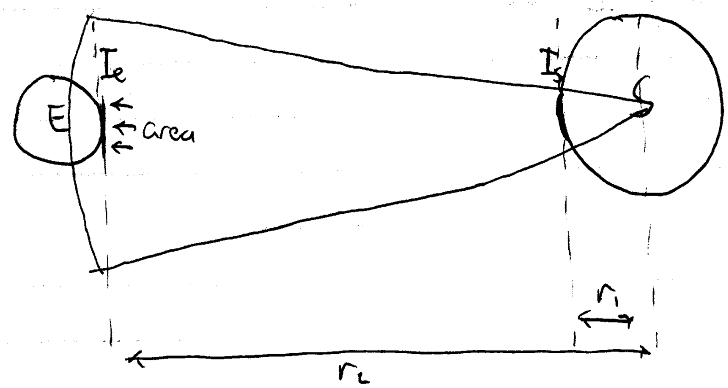
$$\langle \vec{P} \rangle = \frac{I}{c}\hat{k}$$

Since the momentum transfer to a reflector is twice that of the incoming light, the average pressure exerted on the reflector by the sun's radiation is

$$2 \cdot \frac{1.37\ kW/m^2}{3\times10^8\ m/s} = 2 \times 4.57 \frac{N\cdot m}{m^2 s} \cdot \frac{s}{m} \times 10^{-6}$$

$$= 9.14\times10^{-6} \frac{N}{m^2} = 9.14 \frac{\mu N}{m^2}$$

(ii)



Intensity is power/area. The sun emits light onto the surface of a sphere, whose area $\propto r^2$

$\therefore I \propto \frac{1}{r^2}$, i.e. $I = a\frac{1}{r^2}$, with a constant

$\therefore I_s = \frac{a}{r_1^2}$, $I_e = \frac{a}{r_2^2}$, eliminating a

$$r_2^2 I_e = I_s r_1^2$$

since $r_2 = 1.5 \times 10^{11}\,m$, $r_1 = \frac{1.4\times 10^9}{2}\,m$

$$\therefore I_s = \left(1.37\,\frac{kW}{m^2}\right)\left(1.5\times 10^{11}m\right)^2\left(7\times 10^8 m\right)^{-2}$$
$$= 6.3\times 10^7\,\frac{W}{m^2}$$

So $P_s = \frac{I_s}{c} = 0.21\,\frac{N}{m^2}$

3 $n = C_1 + C_2/\lambda^2 + C_4/\lambda^4 + \ldots$

i) consider a dispersionless medium (n does not depend on λ)
If this were true, then since $\lambda \neq 0$, $C_2, C_4, \ldots = 0$
then, $n = C_1$ "ordinary"
Thus, C_1 is the "simple" refractive index, often written as n_0.
From the Lorentz model, the medium could be considered dispersionless far from any resonances

ii) let $n = C_1 + \frac{C_2}{\lambda^2}$

$$n(\lambda = 410\,nm) = 1.557 = C_1 + \frac{C_2}{(410)^2}$$
$$n(\lambda = 550\,nm) = 1.547 = C_1 + \frac{C_2}{(550)^2}$$

subtracting these equations gives $0.010 = C_2\left(\frac{1}{(410)^2} - \frac{1}{(550)^2}\right)$
$C_2 = 3783\ nm^2$

solving for C_1 $C_1 = 1.534$

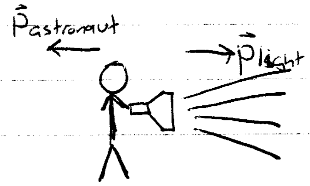
so, with a 2-term Cauchy expansion,
$$n = 1.534 + \frac{3783}{\lambda^2} \quad (\lambda \text{ in nm})$$

$$n(\lambda = 610\,nm) = 1.534 + \frac{3783}{(610)^2} = 1.545$$

iii) I) grad student answer: if you want to know the refractive index of a material at a particular λ, just look it up in a catalogue, don't waste valuable e-mail time trying to calculate things

II) real answer: Cauchy's equation only applies to the visible spectrum. Thus, it's use is limited. Furthermore, it is likely limited to a small bandwidth within the visible spectrum.
In particular, glass is quite transparent in the visible spectrum, but contains electronic resonances in the UV. Thus, C's equation will fail for UV wavelengths

4



Since the flashlight has an "inexhaustable power supply", we assume that the mass of the astronaut + flashlight is constant.

To conserve momentum $|\vec{p}_a| = |\vec{p}_{light}|$ since light is unidirectional $\rightarrow$

$\rightarrow p_a = p_{light}$ (no vector signs needed)

$p_a = m_a v_a$ (of course) $\rightarrow F_a = m_a a$ (no surprise)

$p_{light} = \frac{E_{light}}{c} \rightarrow F_{light} = \frac{P_{light}}{c}$ where P_{light} is the Power in the light beam

Thus, F_{light} causes the astronaut to accelerate

$$\rightarrow m_a a = \frac{P_{light}}{c}$$

$$a = \frac{P_{light}}{m_a c} = \frac{10W}{100kg \cdot 3\times10^8 \frac{m}{s}} = 3.3\times10^{-10} s^{-2}$$

The astronaut starts at rest, so his velocity at any time t after turning on the flashlight is

$$v_a = at$$

$$10 m s^{-1} = 3.3\times10^{-10} s^{-2}\ t$$

$$t = 3^{10} s \approx 951 \text{ years !}$$

5 The average power output of the antenna is calculated like this:

$$\cancel{P_{avg} = 200 kW \times \frac{1}{500} s^{-1} \times 2\times10^{-6}}$$

~~energy~~ power output/sec $= 200 kW \times 500 \frac{pulses}{sec} = 10^8 \frac{W}{sec}$

but each pulse lasts $2 \mu sec$

$$\therefore P_{avg} = 10^8 \frac{W}{sec} \times 2\times10^{-6} sec = 200W$$

$$\therefore I_{avg} = \frac{200W}{(2m)^2} = 50 \frac{W}{m^2}$$ (assuming area of dish $\approx (2m)^2$)

So, pressure $= \frac{50 \frac{W}{m^2}}{c} = 1.67\times10^{-7} \frac{N}{m^2}$

and the total force is $F = PA = 1.67 \times 10^{-7} \frac{N}{m^2} \cdot 4m^2$
$= 6.67 \times 10^{-7} N$

This is a very small force so it shouldn't be significant in antenna design.

6. Since the middle polarizer rotates through a full 2π, this problem has complete rotational symmetry ~~above~~ around the ~~beam's~~ axis given by the beam's direction.

The polarization of the incident beam is not given. Since we have rotational symmetry, ROTATE the entire apparatus (3 polarizers) so that the first polarizer is along the direction of polarization of the beam. Now, DEFINE this to be the x-axis.

Note we can also define the origin of time as we choose. I will choose the rotating polarizer to be an x-polarizer at $t=0$.

Using matrix methods, we now have an x-polarizer, a y-polarizer and a rotating polarizer defined as follows:

$$M_{x\text{-pol}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{y\text{-pol}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{rot} = \begin{pmatrix} \cos^2\omega t & \sin\omega t\cos\omega t \\ \sin\omega t\cos\omega t & \sin^2\omega t \end{pmatrix}$$

(more on this later)

Start with an initial $\vec{E}$ field $\vec{E}_i = \begin{pmatrix} a \\ 0 \end{pmatrix}$

$$\therefore \vec{E}_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2\omega t & \sin\omega t\cos\omega t \\ \sin\omega t\cos\omega t & \sin^2\omega t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ a\sin\omega t\cos\omega t \end{pmatrix}$$

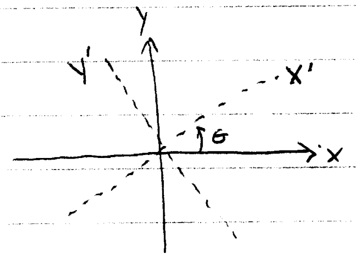
So, the final $I_f \propto |\vec{E}_f|^2 = a^2 \sin^2\omega t\cos^2\omega t = a^2 \cdot \frac{1}{4}\sin^2 2\omega t$

but $1 - 2\sin^2 2\omega t = \cos 4\omega t \rightarrow \sin^2 2\omega t = \frac{1-\cos 4\omega t}{2}$

$\therefore I_f \propto a^2 \cdot \frac{1}{8}(1-\cos 4\omega t)$ and since $I_0 \propto a^2$, then

$$I_f = I_0 \cdot \frac{1}{8}(1 - \cos 4\omega t)$$

How did we get M_{rot}?



Consider a Jones vector $E_0 = \begin{pmatrix} a \\ b \end{pmatrix} = a\hat{x} + b\hat{y}$ ← Electric field vector in x, y basis

We need to write E_0 in the x', y' basis

Clearly, the transformation between co-ordinate systems is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos\theta + b\sin\theta \\ -a\sin\theta + b\cos\theta \end{pmatrix}$$

Now, consider a linear polarizer with it's transmission axis defined as x' (i.e angled at θ w.r.t. x-axis)

After this polarizer, the field vector is

$$\begin{pmatrix} a' \\ b' \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a\cos\theta + b\sin\theta \\ 0 \end{pmatrix}$$

However, we want to work in the x-y basis, so we transform from x'-y' back to x-y.

$$\begin{pmatrix} a \\ b \end{pmatrix}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}_1 = \begin{pmatrix} a\cos^2\theta + b\sin\theta\cos\theta \\ a\sin\theta\cos\theta + b\sin^2\theta \end{pmatrix}$$

So we see that $\begin{pmatrix} a \\ b \end{pmatrix}_1 = M_{rot} \begin{pmatrix} a \\ b \end{pmatrix}$ with $M_{rot} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$

and replacing $\theta = \omega t$ gets us the desired answer

7 i) From Hecht, p. 367, we see that

$\vec{E}_1 = (1,1,0,0) \longrightarrow (1,0) \longrightarrow$ horizontally polarized light

Stokes vector $\qquad$ Jones vector

$\vec{E}_2 (3,0,0,3) \longrightarrow \frac{3}{\sqrt{2}}(1,-i) \longrightarrow$ right handed circularly polarized light

Note that intensity of E_1 is $= 1$ and $E_2 = 9$ $\quad$ (i.e. $I = S_0^2$)

ii) Since the two beams are incoherent, they can be added just like ordinary 4-vectors

$$\vec{E}_3 = \vec{E}_1 + \vec{E}_2$$
$$= (4, 1, 0, 3)$$

So, we have an ellipse of flux density 4, more horizontally polarized than vertical, and right-handed

iii) the degree of polarization is

$$V = (S_1^2 + S_2^2, S_3^2)^{\frac{1}{2}} / S_0$$

$$= \frac{(1^2 + 0^2 + 3^2)^{\frac{1}{2}}}{4} = \frac{\sqrt{10}}{4}$$

8. Let's take a 2-term Cauchy expansion

$$n(\lambda) = C_1 + \frac{C_2}{\lambda^2}$$

Proceeding similarly to q.3

$$1.65338 = C_1 + \frac{C_2}{(490)^2}$$
$$1.62425 = C_1 + \frac{C_2}{(620)^2}$$

$$C_2 = 18631$$
$$C_1 = 1.57578 \text{ nm}^2$$

Thus $n(\lambda = 555nm) = 1.57578 + \frac{18631}{(555)^2} = 1.63627$

Hence $V_{phase} = \frac{c}{n} = \frac{c}{1.63627} = 0.61115\,c \approx 1.83 \times 10^8 \frac{m}{s}$

$V_{group} = V_{phase}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$

What is $\frac{dn}{dk}$? , well $\frac{dn}{dk} = \frac{dn}{d\lambda}\cdot\frac{d\lambda}{dk}$

trivial $\lambda = \frac{2\pi}{k}$ $\therefore d\lambda = -\frac{2\pi}{k^2}dk$

so $\frac{d\lambda}{dk} = -\frac{2\pi}{k^2} = -\frac{2\pi}{(2\pi/\lambda)^2} = \frac{-\lambda^2}{2\pi}$

and $\frac{dn}{d\lambda} = -2\frac{C_2}{\lambda^3}$ (from the Cauchy equation)

so,

$$V_g = V_p\left(1 - \left(\frac{2\pi}{\lambda}\cdot\frac{1}{n}\right)\left(-2\frac{C_2}{\lambda^3}\right)\left(-\frac{\lambda^2}{2\pi}\right)\right) = V_p\left(1 - \frac{2C_2}{n\lambda^2}\right)$$

$$= V_p\left(1 - \frac{2\cdot 18631}{1.63627\cdot(555)^2}\right)$$

$$= V_p(1 - 0.0739)$$

$$= 0.926\,V_p = 0.566\,c$$

(continued on next page)

$$= 1.70 \times 10^8 \frac{m}{s}$$

9 Skin depth $y = \frac{1}{\alpha} = \left(\frac{4\pi K}{\lambda}\right)^{-1}$ with K defined using the complex refractive index of copper, i.e. $\hat{n} = n + iK$

we want $y >$ thickness of one atom $\approx 1\text{Å} = 10^{-10}m$

$\therefore \frac{\lambda}{4\pi K} > 10^{-10}m$

$\therefore \lambda > 10^{-10}\cdot 4\pi K$

since $c = f\lambda \rightarrow \frac{c}{f} > 10^{-10}\cdot 4\pi K$

$f < (10^{-10}\,4\pi K)^{-1} c$

$f < 10^{10} \times \frac{c}{4\pi K}$

8 (continued) Michelson found $V_G = \frac{1}{1.758} c = 0.569c$
we found $V_G = 0.566\ c$

$n(\text{orange-red}) = 1.62425$, $n(\text{green-blue}) = 1.65338$

$\therefore V_p = \frac{1}{1.62425} c = .61567\ c$

$V_p = \frac{1}{1.65338} c = 0.60482\ c$

$\therefore$ orange red is 1.79% faster than green-blue

Furthermore,

$V_g(\text{orange-red}) = 0.941 V_p = 0.579\ c$, $V_g(\text{green-blue}) = 0.906\ V_p = 0.548c$

a 5.65% difference!

10 ~~We can use the constitutive relations:~~

$$\cancel{\vec{D} = \epsilon\vec{E} \quad \text{with } \epsilon = \epsilon_0(1+\chi_e)}$$
$$\cancel{\vec{B} = \mu\vec{H} \quad \mu = \mu_0(1+\chi_m)}$$

a) In vacuum, $\chi_e = \chi_m = 0$

There's a few ways of doing this but the easiest (mathematically) is to write $\vec{E}, \vec{H}$ in phasor notation:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp(i(\vec{k}\cdot\vec{r} - \omega t)) \quad , \vec{E}_0 = E_0 \hat{r}$$
$$\vec{H}(\vec{r},t) = \vec{H}_0 \exp(i(\vec{k}\cdot\vec{r} - \omega t))$$

In this notation $\vec{\nabla} \leftrightarrow i\vec{k}$ and $\frac{\partial}{\partial t} \leftrightarrow -i\omega$

Then Faraday and Ampère's laws:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\mu\vec{H})$$
$$\vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t}(\epsilon\vec{E})$$

can be written as

$$i\vec{k} \times \vec{E}_0 = i\omega\mu\vec{H}_0$$
$$i\vec{k} \times \vec{H}_0 = -i\omega\epsilon\vec{E}_0$$

We now cross-multiply Faraday's law by $i\vec{k}$ and substitute Ampère's law

$$i\vec{k}\times(i\vec{k}\times\vec{E}_0) = -\omega\mu\,\vec{k}\times\vec{H}_0$$

$$-\vec{k}\times(\vec{k}\times\vec{E}_0) = -\vec{k}(\vec{k}\cdot\vec{E}_0) + k^2\vec{E}_0 = \omega^2\mu\epsilon\vec{E}_0$$

expanding the triple product

but, $\vec{k}\cdot\vec{E}_0 = 0$ (Gauss' law in a charge-free medium)

$$\therefore\quad k^2\vec{E}_0 = \omega^2\mu\epsilon\vec{E}_0$$

so $\boxed{k^2 = \omega^2\mu\epsilon}$ dispersion relation of plane wave

now, back to Ampère's law: $i\vec{k}\times\vec{E}_0 = i\omega\mu\vec{H}_0$

$$\therefore\quad \vec{H}_0 = \frac{1}{\omega\mu}\,\vec{k}\times\vec{E}_0$$

using ~~[illegible]~~ the dispersion relation $\omega = k\cdot\frac{1}{\sqrt{\mu\epsilon}}$

$$\therefore\quad \vec{H}_0 = \sqrt{\frac{\epsilon}{\mu}}\,\frac{\vec{k}}{k}\times\vec{E}_0 = \sqrt{\frac{\epsilon}{\mu}}\,\hat{k}\times\vec{E}_0$$

, with $\hat{k}$ being a unit vector in the $\vec{k}$-dir.

This makes it clear that $\dfrac{|\vec{E}_0|}{|\vec{H}_0|} = \sqrt{\dfrac{\mu}{\epsilon}}$

In vacuum, $\epsilon = \epsilon_0$, $\mu = \mu_0$, so $\dfrac{|\vec{E}_0|}{|\vec{H}_0|} = Z_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}}$

$$= 377\,\Omega$$

For a medium of refractive index n, $V_\phi = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ (vacuum) $\rightarrow \frac{1}{n\sqrt{\mu_0\epsilon_0}}$ (material)

$$\therefore\quad Z = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu\epsilon}{\epsilon\epsilon}} = \frac{1}{\epsilon}\sqrt{\mu\epsilon} = \frac{1}{\epsilon}\frac{1}{V_\phi} = \frac{1}{\epsilon}\frac{n}{c}$$

but $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, $\therefore\ \frac{E}{H} \rightarrow \frac{1}{\epsilon}\,n\sqrt{\mu_0\epsilon_0\times\frac{\epsilon_0}{\epsilon_0}} = \frac{\epsilon_0}{\epsilon}\,n\sqrt{\frac{\mu_0}{\epsilon_0}}$

$$Z = \frac{\epsilon_0}{\epsilon}\,n Z_0$$