Lecture 24

Gaussian Beam Optics

Perturbation Analysis

- Many of the examples we have treated have phase which
  - varies fast in the DOP (z)
  - varies slowly as a function of the other co-ordinates (x,y)
- Fresnel diffraction varies quadratically in the x,y direction
  - (1/r)exp(ikr) -> (1/z)exp(ikz) exp(ik(x^2+y^2)/(2z))
  - separates the fast-changing z-direction from the slow changing x,y direction
- We can generalise!
- write the wave as u(x,y,z)exp(ikz) - subst in wave equation

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik\frac{\delta u}{\delta z} = 0$$

- ∇_T is the 2-D grad function

Paraxial Wave Equation

- If we now assume that the function
  - varies slowly in z on the scale of a wavelength $|\partial u/\partial z| \ll k|u|$
  - that it is smooth - higher order differentials can be ignored

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik\frac{\delta u}{\delta z} \rightarrow \nabla_T^2 u + 2ik\frac{\delta u}{\delta z} = 0$$

- The *slowly varying envelope approximation (SVEA)* leads to the *paraxial wave equation*
- Note that $u(x,y,z)=(1/z)\exp(ik(x^2+y^2)/(2z))$ is a solution of the above equation
- Trouble with that solution is that it has infinite extent - is there a similar solution which has finite extent - ie looks something like a pencil beam?

A Gaussian Beam

- Yes, there is....
  - $(1/z)\exp(ik(x^2+y^2)/(2z))$ ->
  - $u_{00}'(x,y,z) = (1/(z-iz_0))\exp(ik(x^2+y^2)/(2(z-iz_0))$
  - $z \to z - iz_0$
- Normalise over x, y

$$u_{00}(x,y,z) = \sqrt{\frac{kz_0}{\pi}}\,\frac{1}{z-iz_0}\,\exp\left(\frac{ik[x^2+y^2]}{2(z-iz_0)}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{[x^2+y^2]}{w^2}\right)\exp\left(\frac{ik[x^2+y^2]}{2R}\right)$$

- $w^2(z) = w_0^2(1 + z^2/z_0^2)$, $\quad w_0^2 = 2z_0/k$
- $R = (z^2+z_0^2)/z$, $\quad \tan\varphi = z/z_0$

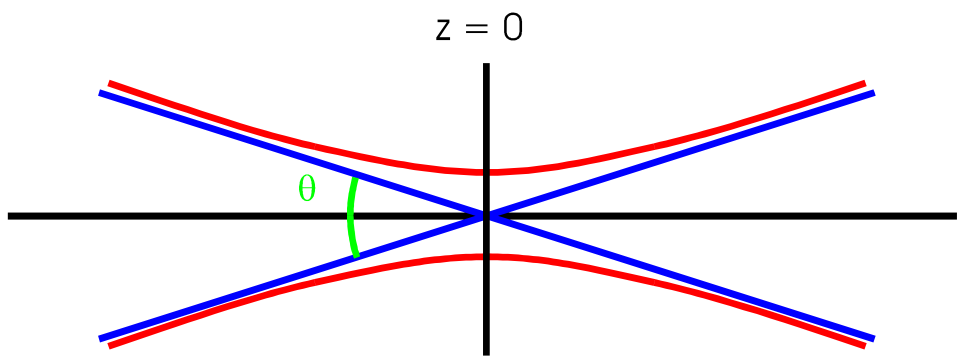
Properties of the Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{r^2}{w^2}\right)\,\exp\left(\frac{ikr^2}{2R}\right)$$

- *Fundamental Gaussian Beam Solution*
- One parameter - z_0 - describes the beam
- Circularly symmetric - function of r
- Gaussian extent transversely - w is the e^{-1} point of amplitude
- at z = 0, w = w_0 and is the minimum extent of beam
- at z = z_0, w = $\sqrt{2}\, w_0$ - z_0 is called the *confocal parameter*

Properties of the Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\frac{1}{w}\, e^{-i\varphi}\, \exp\left(-\frac{r^2}{w^2}\right) \exp\left(\frac{ikr^2}{2R}\right)$$



- What is the significance of R?
- R is the radius of curvature of the surfaces of constant phase
- as z → 0, R → ∞
- as z → ∞, R → z
- What is the significance of φ?
  - φ is related to the velocity of surfaces of constant phase
  - Remember this isn't a plane wave and doesn't even satisfy the full wave equation

Any Other Solutions?

- Yes, it can be shown that...

$$u_{l,m}(x,y,z) =$$

$$\frac{C_{l,m}}{w} H_l\left[\frac{x\sqrt{2}}{w}\right] H_m\left[\frac{y\sqrt{2}}{w}\right] e^{-i(l+m+1)\varphi} \exp\left(-\frac{r^2}{w^2}\right)\exp\left(\frac{ikr^2}{2R}\right)$$

- Where $H_m(x)$ is a Hermite polynomial
- $C_{l,m}$ is a normalisation constant
- l and m are integers
- AKA TEM_{lm}
- Phase speed is

$$k_{eff} = k - (l + m + 1)\frac{z_0}{z^2 + z_0^2}$$

Paraxial Optics With Gaussian Beams

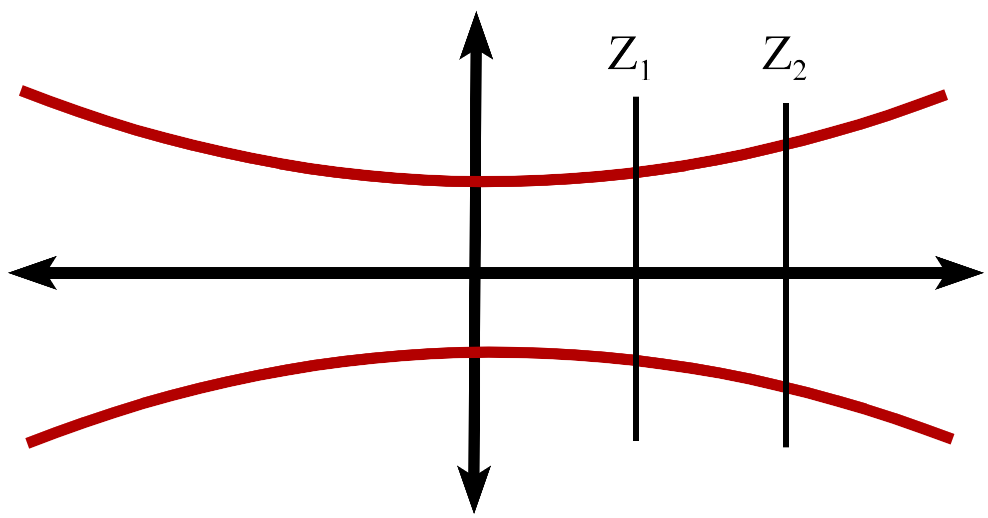
- A Gaussian beam is a function of z_0
- Gaussian beam at any position is a function of $q = z - i z_0$
- A Gaussian beam is locally a spherical wave with radius of curvature R
- Consider a Gaussian beam on a thin lens
- The size of the beam doesn't change on either side of the lens
- The curvature of the beam is altered by the lens
- IT CAN BE SHOWN THAT
  - The transformation of the parameter q is given by
  - $q` = (Aq + B)/(Cq + D)$
  - where A,B,C,D are the elements of the 2x2 matrix for paraxial ray optical systems
  - Notice that this is NOT a matrix equation

Transformations of Gaussian Beams

- Stated that $u(x,y,z)=(1/q)\exp(ik(x^2+y^2)/(2q))$ is a solution to the paraxial wave equation
- Where $q = z - i\, z_0$, z is the distance in the DOP and z_0 describes the Gaussian beam (Gaussian beam parameter)
- By inspection of wave equation solution show q(z) propagates as:
- $1/q(z) = 1/R(z) + i\ \lambda/(\pi\, n\, w(z)^2)$
- Far field propagate like spherical wave $R(z) \to z$, $z \to \infty$

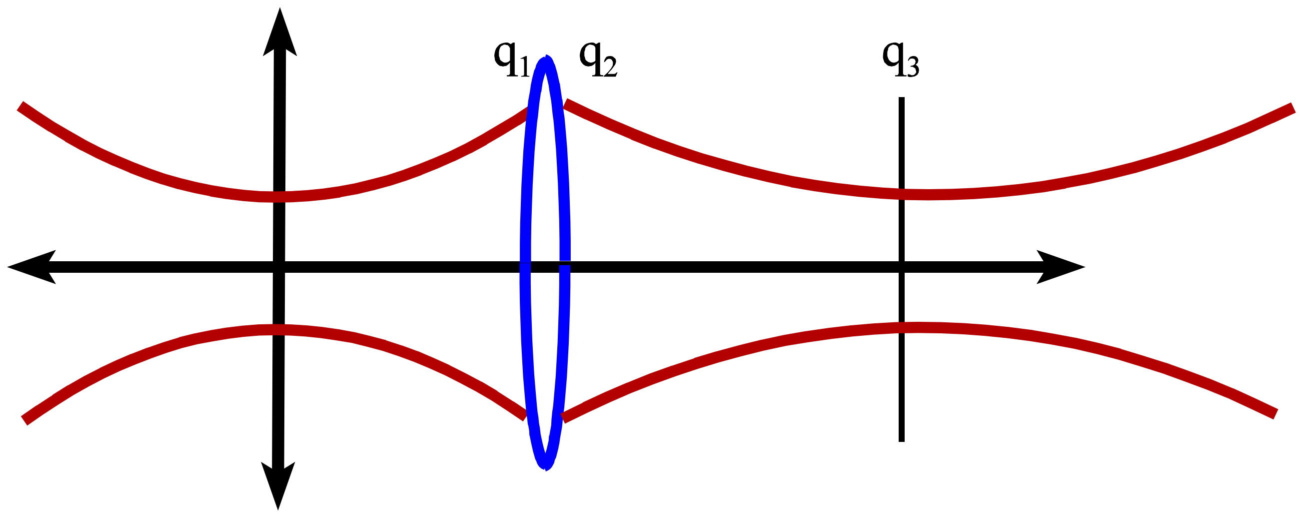
Free Space Gaussian Beam

- Consider a Gaussian beam propagating in free space
- Reference frame for Gaussian beams defines waist at $z=0$
- At plane $z=z_1$ we can write $q_1 = z_1 - i\, z_0$
- At plane $z=z_2$ we can write $q_2 = z_2 - i\, z_0 = q_1 + (z_2 - z_1)$
- Plane shift corresponds to change in real component of q
- Can calculate beam width and curvature at new position as



$$1/q(z) = 1/R(z) + i\, \lambda/(\pi\, n\, w(z)^2)$$

Gaussian Beams and Optics



- Consider a thin lens with an incident Gaussian Beam
- Spot size does not change as beam passes through lens
- But, lens changes wave front (earlier Fraunhoffer lecture)

$$T(x,y) = \exp\left(-ik(x^2+y^2) / 2f \right)$$

- Apply this directly to wave equation solution at lens boundary => effect is to change R->R'

$$\frac{ik(x^2+y^2)}{2R'} = \frac{ik(x^2+y^2)}{2R} - \frac{ik(x^2+y^2)}{2f}$$

Gaussian Beams and Optics II

- We can simplify this as 1/R’ = 1/R - 1/f
- So, 1/q’ = 1/q - 1/f [the spot size does not change]
- We can therefore write the lens transformation as

$$q' = q / (-q/f + 1)$$

- Equivalent to defining a new origin and Gaussian beam parameter for the propagating wave
- As previously outlined, the general transformation is

$$q` = (Aq + B)/(Cq + D)$$

- Known as a fractional linear or Möbius transformation
- To confuse matters further the transformation is also called the ABCD law and is written as a pseudo-matrix [A B; C D]
- THIS HAS **NOTHING** TO DO WITH MATRIX ALGEBRA!

Common Transformations

- Free Space Transformation becomes [1 z_2-z_1; 0 1]
- Thin lens with focal length f becomes [1 0; -1/f 1]
- Note that converging lenses have +ve f by convention
- Series of optical components can be represented by result of successive transformations in q space

$$M_{series} = M_1 \ldots M_{last}$$

- Multiple transformations are applied in an order determined by the beam propagation

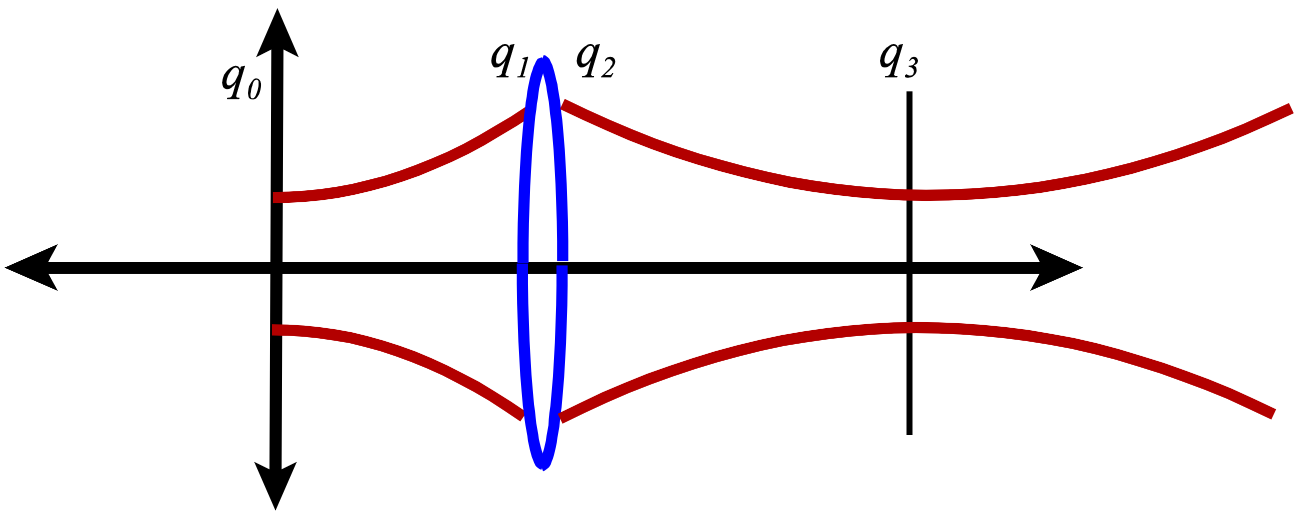
Gaussian Beam Summary

- A Gaussian beam is the paraxial limit solution to a wave equation
- Equivalent to a point source shifted by an imaginary amount iz_0
- With $z_0 = 0$ paraxial solution is just portion of spherical wave emanating from the origin
- $q = z - i z_0$ measures complex distance from reference plane (waist of Guassian beam)
- Transforms in q-space can be used to model effect of optics on a Gaussian beam

Applications of Gaussian Beam Optics

- Results are remarkably similar to ray tracing but equations include wave behaviour of light
- Gaussian beam does not behave quite like a plane wave and consequently equivalent calculations give slightly different answers
- Laser people routinely carry out these calculations
- Simple example next lecture...

Gaussian Beam Optics: Example



Question: A He-Ne laser operating in Gaussian mode has a divergence of 1mR and with a beam waist at the output of 0.4mm. What is the diffraction limited spot size that we can achieve with a lens f=+2cm located 1m from the beam waist?