Lecture 23

Gaussian Beams
Higher Orders and Modes

Paraxial Wave Equation

- If we now assume that the function
  - varies slowly in z on the scale of a wavelength
    $|\partial u/\partial z| \ll k|u|$
  - that it is smooth - higher order differentials can be ignored

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik\frac{\delta u}{\delta z} \rightarrow \nabla_T^2 u + 2ik\frac{\delta u}{\delta z} = 0$$

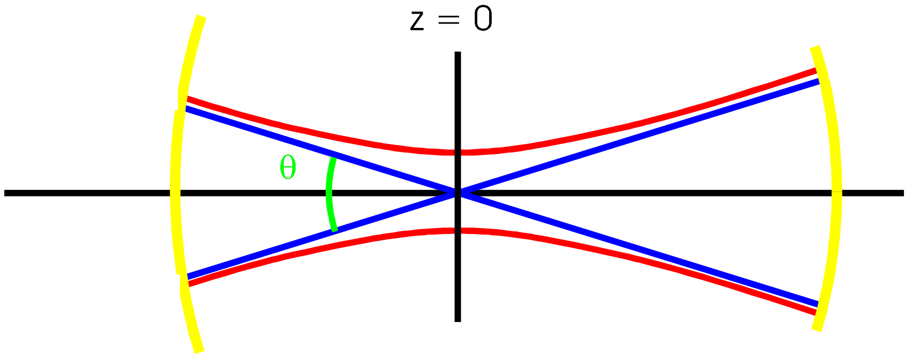
- The *slowly varying envelope approximation (SVEA)* leads to the *paraxial wave equation*

A Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\exp\left(-\frac{[x^2+y^2]}{w^2}\right)\exp\left(\frac{ik[x^2+y^2]}{2R}\right)$$

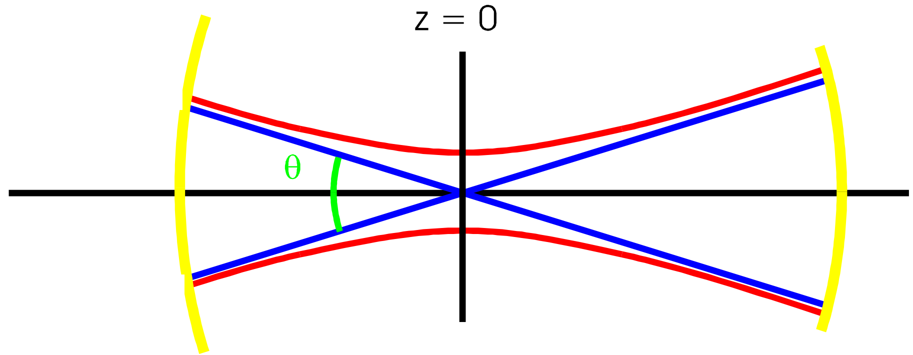
- $w^2(z) = w_0^2(1 + z^2/z_0^2)$, $\quad w_0^2 = 2z_0/k$
- $R = (z^2+z_0^2)/z$, $\quad \tan\varphi = z/z_0$
- *Fundamental Gaussian Beam Solution*
- Function of one parameter - z_0
- Circularly symmetric - function of r
- Gaussian extent transversely - w is the e^{-1} point of amplitude
- at z = 0, $w = w_0$ and is the minimum extent of beam
- at $z = z_0$, $w = \sqrt{2}\, w_0$ - z_0 is called the *confocal parameter*

Standing Waves In a 1.5-D cavity



- A cavity is formed by two mirrors assumed for simplicity to be 100% reflecting
- A mode (or standing wave) happens if...
  - Amplitude of field at any point is stationary
  - Phase is also stationary
- Neglect refraction losses
- Can set up a mode if beam is Gaussian and mirrors match R_1,R_2 the radii of the wavefronts at each end

Standing Waves In a 1.5-D cavity



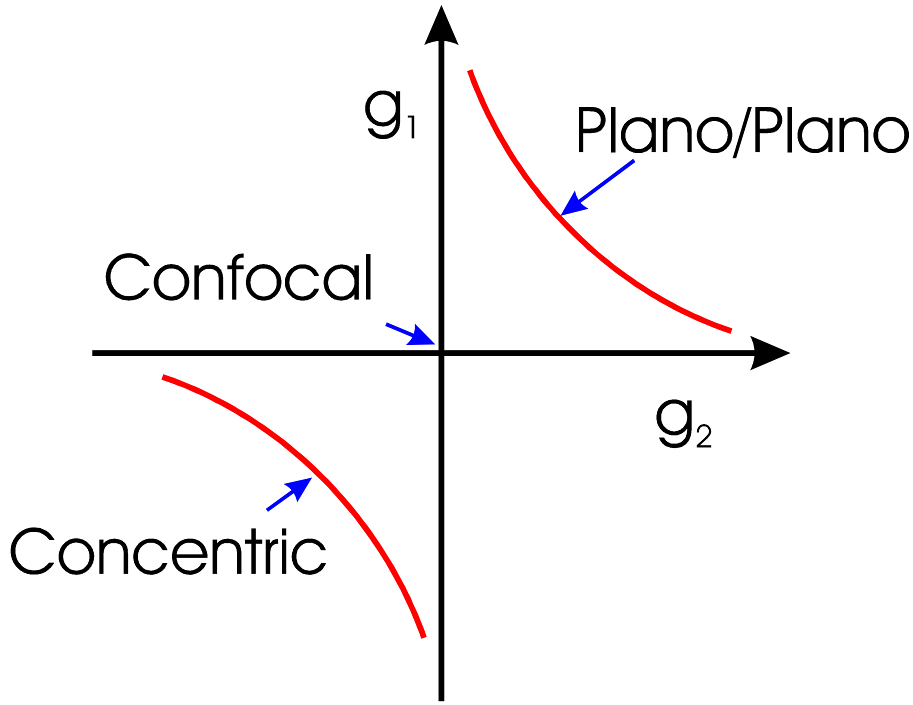
- and for the spot sizes on the mirror $w_{1,2}$

$$w_{1,2}^2 = \frac{L\lambda}{\pi}\sqrt{\frac{g_{2,1}}{g_{1,2}(1-g_1 g_2)}}$$

- These solutions exist if $0 < g_1 g_2 < 1$
- Stable solutions require all of w_1, w_2 and w_0 to be finite and larger than the wavelength λ

Standing Waves In a 1.5-D cavity

- If $g_1 = g_2 = 1$ both mirrors are plane - FP
  - all w are infinite!
- If $g_1 = g_2 = 0$, R = L and the cavity is confocal
  - OK but $w_0 = 0$
- If $g_1 = g_2 = -1$, R = L/2 the cavity is concentric
  - $w_{1,2}$ are infinite!



Any Other Solutions?

- Yes, it can be shown that...

$$u_{l,m}(x,y,z) = \frac{C_{l,m}}{w} H_l\left[\frac{x\sqrt{2}}{w}\right] H_m\left[\frac{y\sqrt{2}}{w}\right] e^{-i(l+m+1)\varphi} \exp\left(-\frac{r^2}{w^2}\right)\exp\left(\frac{ikr^2}{2R}\right)$$

- Where $H_m(x)$ is a Hermite polynomial
- $C_{l,m}$ is a normalisation constant
- l and m are integers
- AKA TEM_{lm}
- Phase speed is

$$k_{eff} = k - (l + m + 1)\frac{z_0}{z^2 + z_0^2}$$

Phase Speed to Resonator Frequencies

- If the total phase change through a cavity "trip" is 2qπ, then mode is stationary and cavity "resonates"

$$\int_{z_1}^{z_2} k_{eff}\, dz = q\pi$$

$$= k(z_2 - z_1) - (l + m + 1)\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right] + \frac{\varphi_1 + \varphi_2}{2}$$

- where the φ refer to phase changes on the mirrors which we will now ignore

Phase Speed to Resonator Frequencies

- and if the frequency is related to k by ωn/c

$$\omega_q^{l,m} = \frac{q\pi c}{nL} + \frac{c}{nL}(l + m + 1)\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right]$$

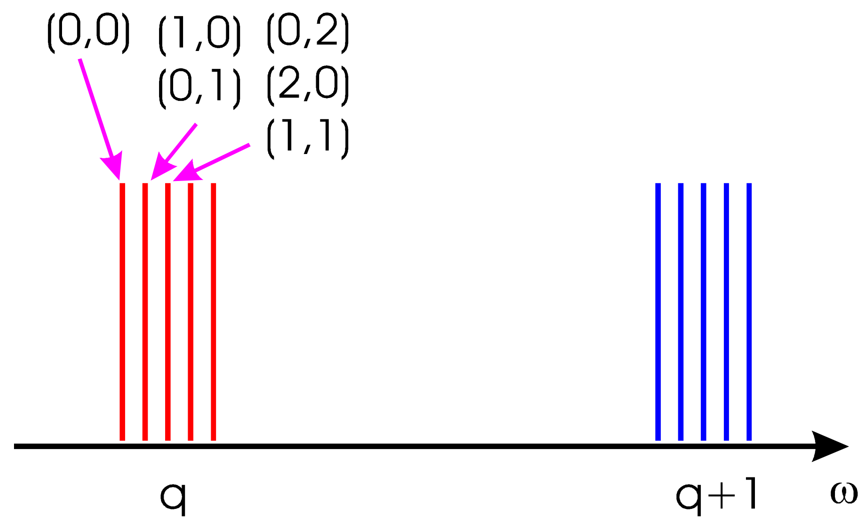
$$= \frac{c\pi}{nL}\left[q + \frac{(l + m + 1)\cos^{-1}\sqrt{g_1 g_2}}{\pi}\right]$$

- Two conditions are useful
  - Near planar for which g_1, g_2 are about 1
  - Near confocal for which g_1, g_2 are about 0

Near-Planar Cavities

- $\cos^{-1}(g_1 g_2)^{1/2} = \alpha \ll \pi$

$$\omega_q^{l,m} = \frac{c\pi}{nL}\left[q + \frac{(l+m+1)\,\alpha}{\pi} \right]$$

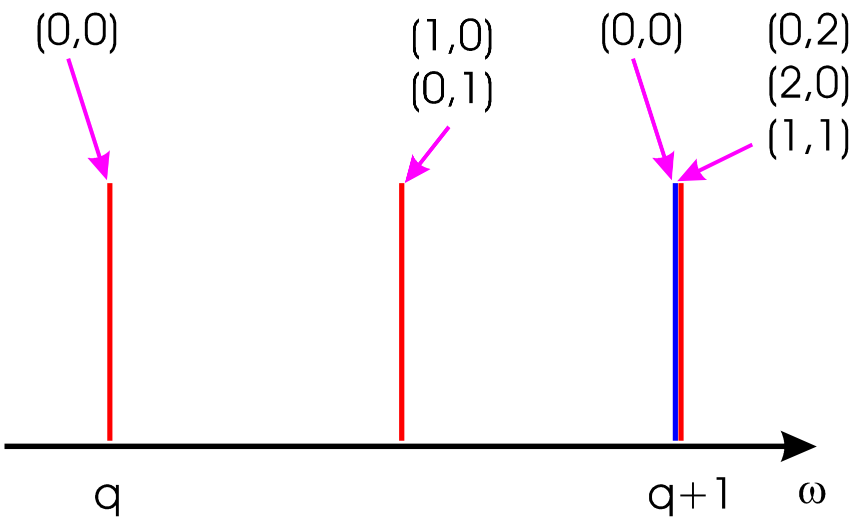


- Modes spread all over the place!!
- Many closely spaced modes
- Lowest l+m is lowest frequency
- Spacing in l+m is much less than spacing in q
  - q is the axial mode number
  - l,m are the transverse mode numbers
- when α goes to zero all l,m modes become degenerate
  - only depends upon q

Near-Confocal Cavities

- $\cos^{-1}(g_1 g_2)^{1/2} = \pi/2$

$$\omega_q^{l,m} = \frac{c\pi}{nL}\left[q + \frac{(l+m+1)}{2}\right]$$



- Modes tightly spaced
- Lowest l+m is lowest frequency
- Spacing in l+m is half the spacing in q
  - q is the axial mode number
  - l,m are the transverse mode numbers
- There is degeneracy of modes, but only at discrete frequencies

Paraxial Optics With Gaussian Beams

- A Gaussian beam is a function of z_0
- Gaussian beam at any position is a function of $q = z - i z_0$
- A Gaussian beam is locally a spherical wave with radius of curvature R
- Consider a Gaussian beam on a thin lens
- The size of the beam doesn't change on either side of the lens
- The curvature of the beam is altered by the lens
- IT CAN BE SHOWN THAT
  - The transformation of the parameter q is given by
  - $q` = (Aq + B)/(Cq + D)$
  - where A,B,C,D are the elements of the 2x2 matrix for paraxial ray optical systems
  - Notice that this is NOT a matrix equation