Lecture 22

Gaussian Beams
Paraxial or Perturbation Analysis
Perturbation Analysis

- Many of the examples we have treated have phase which
  - varies fast in the DOP (z)
  - varies slowly as a function of the other co-ordinates (x,y)
- Fresnel diffraction varies quadratically in the x,y direction
  - \((1/r)\exp(ikr) \rightarrow (1/z)\exp(ikz) \exp(ik(x^2+y^2)/(2z))\)
  - separates the fast-changing z-direction from the slow changing x,y direction
- Can we generalise? (Yes, or I wouldn’t be doing this...)
- write the wave as \(u(x,y,z)\exp(ikz)\) - subst in wave equation

\[
\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik \frac{\delta u}{\delta z} = 0
\]

- \(\nabla_T\) is the 2-D grad function
If we now assume that the function

- varies slowly in $z$ on the scale of a wavelength
  $$|\partial u/\partial z| \ll k|u|$$
- that it is smooth - higher order differentials can be ignored

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik \frac{\delta u}{\delta z} \rightarrow \nabla_T^2 u + 2ik \frac{\delta u}{\delta z} = 0$$

The slowly varying envelope approximation (SVEA) leads to the paraxial wave equation

Note that $\left(1/z\right)\exp(ik(x^2+y^2)/(2z))$ is a solution of the above equation

Trouble with that solution is that it has infinite extent - is there a similar solution which has finite extent - ie looks something like a pencil beam?
A Gaussian Beam

- Yes, there is....
  - \(\frac{1}{z} \exp(\frac{ik(x^2+y^2)}{2z})\) ->
  - \(u_{00}(x,y,z) = \frac{1}{(z-iz_0)} \exp(\frac{ik(x^2+y^2)}{2(z-iz_0)})\)
  - \(z \to z - iz_0\)
- Normalise over \(x, y\)

\[
u_{00}(x,y,z) = \frac{\sqrt{kz_0}}{\pi} \frac{1}{z - iz_0} \exp\left(\frac{ik[x^2+y^2]}{2(z - iz_0)}\right)
\]

\[
= \sqrt{\frac{2}{\pi}} \frac{1}{w} e^{-i\phi} \exp\left(-\frac{[x^2+y^2]}{w^2}\right) \exp\left(\frac{ik[x^2+y^2]}{2R}\right)
\]

- \(w^2(z) = w_0^2(1 + z^2/z_0^2),\quad w_0^2 = 2z_0/k\)
- \(R = (z^2+z_0^2)/z,\quad \tan \phi = z/z_0\)
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi w}} \exp \left( -\frac{r^2}{w^2} \right) \exp \left( \frac{ikr^2}{2R} \right) \]

- **Fundamental Gaussian Beam Solution**
- Function of one parameter - \( z_0 \)
- Circularly symmetric - function of \( r \)
- Gaussian extent transversely - \( w \) is the \( e^{-1} \) point of amplitude
- at \( z = 0, w = w_0 \) and is the minimum extent of beam
- at \( z = z_0, w = \sqrt{2} w_0 \) - \( z_0 \) is called the *confocal parameter*
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{1}{w} \ e^{-i\phi} \ \exp\left(-\frac{r^2}{w^2}\right) \ \exp\left(\frac{ikr^2}{2R}\right) \]

- What is the shape of the red line of constant amplitude?
- \[ \frac{r^2}{w^2} = C \]
- \[ r^2 - (Cw_0^2/z_0^2)z^2 = Cw_0^2 \]
- Hyperolas
- Note that the confocal parameter \( z_0 \) gives the distance over which the beam is “sort of” collimated and \( w_0 \) gives the minimum beam size
- Since \( z_0 = kw_0^2/2 \) you can’t have a small beam collimated for a long distance. - small waist, large divergence and vv
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{1}{w} e^{-i\varphi} \exp \left( -\frac{r^2}{w^2} \right) \exp \left( \frac{ikr^2}{2R} \right) \]

- What is the asymptotic angle \( \theta \)?
- for large \( r,z \),
  \[ \tan(\theta/2) = \frac{w}{z} \]
  \[ = \frac{w_0(z/z_0)}{z} \]
  \[ = \frac{w_0}{z_0} = \frac{2}{(kw_0)} \]
- Putting in numbers at 500nm and a 1mm spot \( \theta = 0.32 \text{mr} \)
- ie spot size increases by 1mm/3m - laser-beam like
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{1}{w} \exp\left( -\frac{r^2}{w^2} \right) \exp\left( \frac{ikr^2}{2R} \right) \]

- What is the significance of \( R \)?
  - \( R \) is the radius of curvature of the surfaces of constant phase.
  - As \( z \to 0 \), \( R \to \infty \)
  - As \( z \to \infty \), \( R \to z \)

- What is the significance of \( \varphi \)?
  - \( \varphi \) is related to the velocity of surfaces of constant phase.
  - Remember this isn’t a plane wave and doesn’t even satisfy the full wave equation.
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi w}} \ e^{-i\phi} \ \exp \left( - \frac{r^2}{w^2} \right) \ \exp \left( \frac{ikr^2}{2R} \right) \]

- Beam propagates as \( \exp(ikz-i\phi) \)
- If we take \( k_{\text{eff}} \) as the average over \( 0 \to z \)

\[ k_{\text{eff}} = k - \frac{d\phi}{dz} = \frac{\omega}{c} - \frac{z_0}{Z^2 + z_0^2} < \frac{\omega}{c} \]

- which implies that the phase velocity is > \( c \) which isn’t a problem because this isn’t a solution of the full wave equation
Properties of the Gaussian Beam

\[ u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{1}{w} e^{-i\phi} \exp \left( - \frac{r^2}{w^2} \right) \exp \left( \frac{ikr^2}{2R} \right) \]

- When is all this valid?
- When \(|\partial u/\partial z| \ll k|u|
- which turns out to be
- when \(1/z \ll k\) or
- \(\lambda \ll 1.4\pi w_0\)
- Beam waist much larger than wavelength
A cavity is formed by two mirrors assumed for simplicity to be 100% reflecting.

A mode (or standing wave) happens if...
- Amplitude of field at any point is stationary
- Phase is also stationary
- Neglect refraction losses
- Can set up a mode if beam is Gaussian and mirrors match $R_1, R_2$ the radii of the wavefronts at each end
Standing Waves In a 1.5-D cavity

- \( R_1 = (z_1^2 + z_0^2)/z \)
- \( R_2 = (z_2^2 + z_0^2)/z \)
- Remember one \( z \) has to be -ve
- Let \( z_2 - z_1 = L \)
- Let \( g_{1,2} = (1 - L/R_{1,2}) \)
- Solve for \( z_0 \)

\[
  z_0^2 = \frac{L^2 g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2g_1 g_2)^2},
\]

\[
  w_0^2 = \frac{2}{k} \frac{L \sqrt{g_1 g_2 (1 - g_1 g_2)}}{(g_1 + g_2 - 2g_1 g_2)}
\]
Standing Waves In a 1.5-D cavity

- and for the spot sizes on the mirror \( w_{1,2} \)

\[
w_{1,2}^2 = \frac{L\lambda}{\pi} \sqrt{\frac{g_{2,1}}{g_{1,2}(1 - g_1 g_2)}}
\]

- These solutions exist if \( 0 < g_1 g_2 < 1 \)
- Stable solutions require all of \( w_1, w_2 \) and \( w_0 \) to be finite and larger than the wavelength \( \lambda \)
Standing Waves In a 1.5-D cavity

- If $g_1 = g_2 = 1$ both mirrors are plane - FP
  - all $w$ are infinite!
- If $g_1 = g_2 = 0$, $R = L$ and the cavity is confocal
  - OK but $w_0 = 0$
- If $g_1 = g_2 = -1$, $R = L/2$ the cavity is concentric
  - $w_{1,2}$ are infinite!
Standing Waves In a 1.5-D cavity

- if $g_1 = 0$, $g_2 < 1$ we have one flat mirror at the beam waist and one curved mirror

$$w_{0,1}^2 = \frac{2L}{k} \sqrt{\frac{g_2}{1 - g_2}}, \quad w_2^2 = \frac{2L}{k} \sqrt{\frac{1}{g_2(1 - g_2)}}$$

- This is OK for a range of $g_2$
- For $L = 1\text{m}$, $\lambda = 1\mu\text{m}$ for $R_2 = 5\text{m}-20\text{m}$ $w_{0,1,2}$ of order 1mm