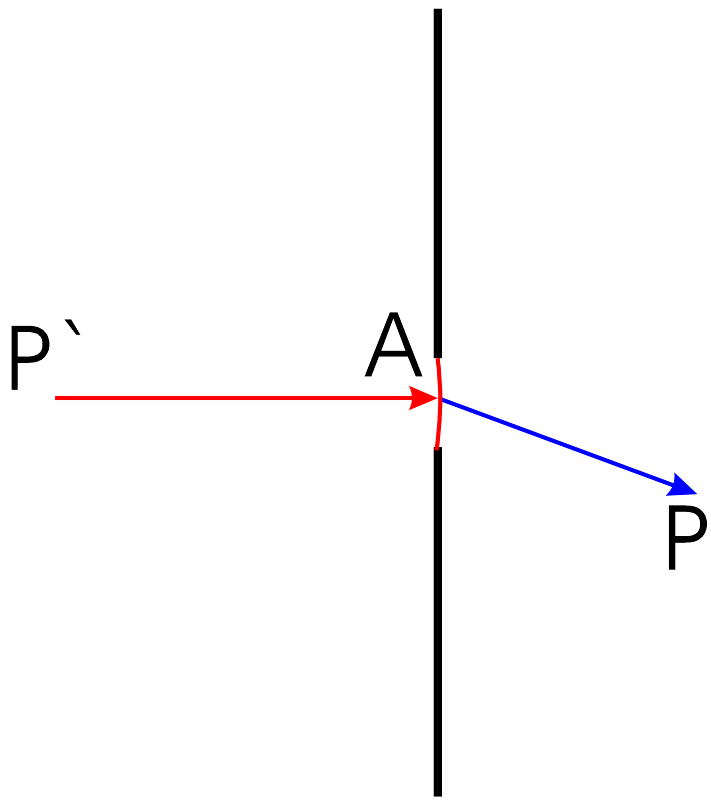
Lecture 19

Fresnel Diffraction - The Twilight Zone
Fraunhoffer Diffraction - Transforms, Transforms, Everywhere

It's Still Too Complicated!!

- Scalar wave equation applied to a screen with a hole in it

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$



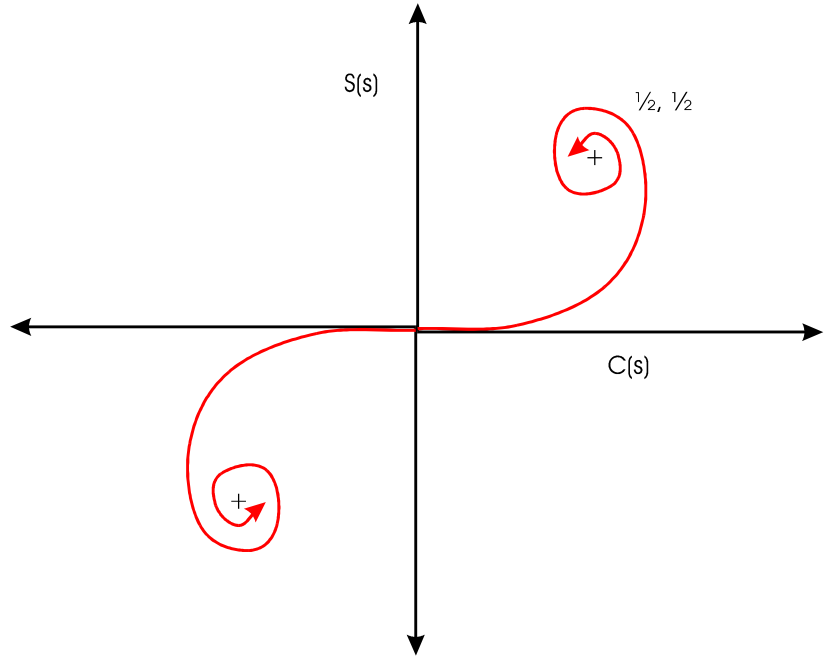
- This is the Fresnel-Kirchoff integral formula
- Shows a phase factor of -i for the diffracted wave
- Integration of secondary radiators across the aperture with an obliquity factor

The Rectangular Slit

$$U_P = B \int_{x_1}^{x_2} \exp\left(\frac{i\pi u^2}{2}\right) dx \int_{y_1}^{y_2} \exp\left(\frac{i\pi v^2}{2}\right) dy$$

$$= \frac{U_{P_0}}{(1+i)^2} \left[C(s) + iS(s)\right]_{x_1}^{x_2} \left[C(s) + iS(s)\right]_{y_1}^{y_2}$$

- If limits are infinity the signal must be U_{P0} [normalization]
- Now take in infinitely long slit
  - eliminates y_0 terms
- Take a single edge at x_0=x

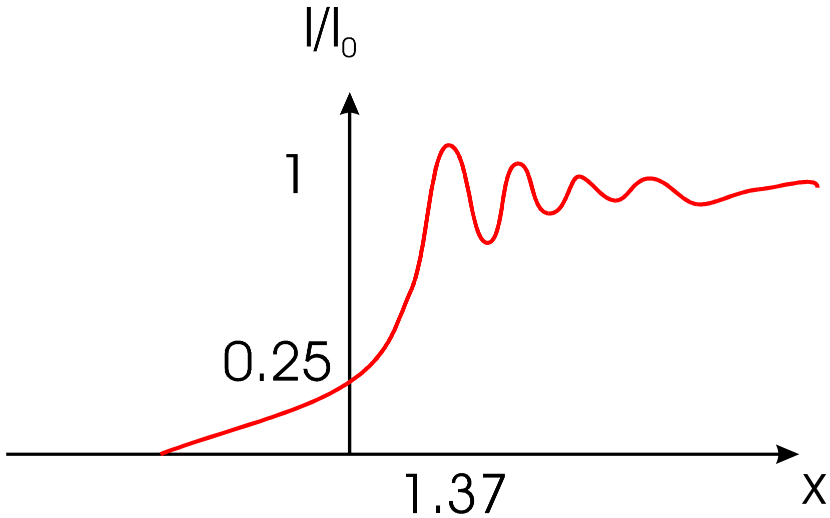


The Fresnel Integral

$$U_P = \frac{U_{P_0}}{(1+i)^2}\,[C(s)+iS(s)]_{x_1}^{x_2}\,[C(s)+iS(s)]_{y_1}^{y_2}$$

$$= \frac{U_{P_0}}{(1+i)}\left(C(x)+iS(x)+\frac{1}{2}+\frac{1}{2}i\right)$$

- At x = 0, $U_P = U_{P0}/2$ which implies 0.25 Intensity
- Moving x is equivalent to moving observation point since everything else is (semi-)infinite



The Circular Aperture

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$

$$= \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} (r_0 - r_m)^2 \right) Q \; dS$$

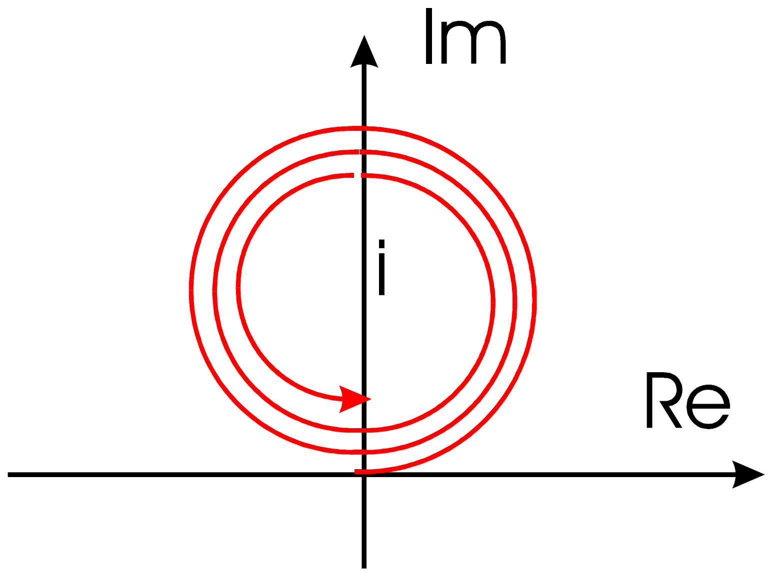
- Need Obliquity Factor to save us from a fate worse than death (an oscillating integral!!)

The Circular Aperture

- If $r_m = 0$ (paraxial approximation), then we can recast this problem in circular symmetry in terms of $\psi = kr_0^2/(2z_a)$ and U_{P0} the undisturbed wave

$$U_P = -iU_{P_0} \int_0^{\psi_0} \exp(i\psi)\ Q(\psi)\ d\psi$$

- $\psi_0 = kR^2/2z$
- This integral oscillates
- If $Q(\psi) \rightarrow 0$ as $\psi \rightarrow \infty$, then it collapses to a value of i
- Useful to consider places where $\mathrm{Re}(\exp(i\psi)) > 0$ separately from areas where $\mathrm{Re}(\exp(i\psi)) < 0$
- Divide integral into zones



Fresnel Zones in the Aperture

- Bright zones $0<\psi<\pi$
- Since ψ is quadratically related to r, the radius $\psi = kr_0^2/(2z_a)$ these regions are also quadratic - but therefore of equal area
- For large numbers of zones - the value of the integral converges to half of the value from the first zone
- Reason is the adjacent zones tend to cancel one another.
- Number of zones is determined by ψ_0

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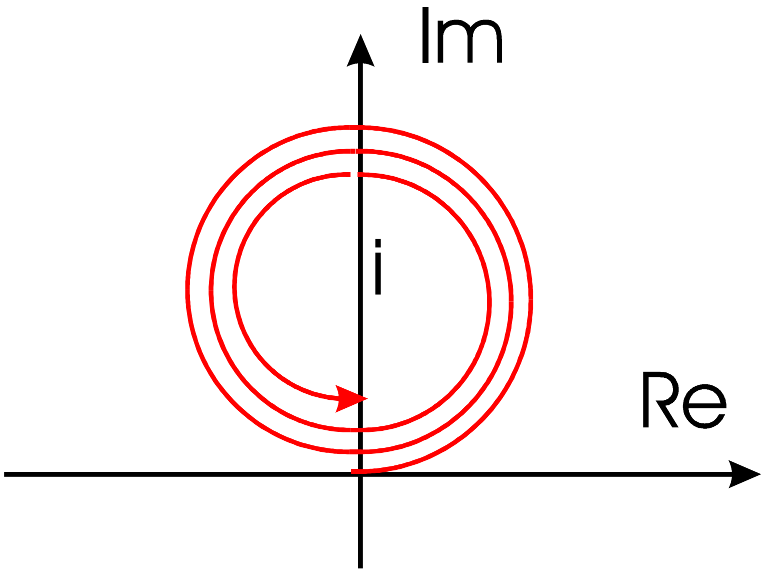
The Fresnel Lens

$$J_P = -iU_{P_0} \int_0^{\psi_0} \exp(i\psi)\ Q(\psi)\ d\psi$$



- If we manipulate the zones by adding an opaque mask to eliminate all negative contributions - the positive contributions add up and a large amount of energy arrives at the focus - a fresnel lens
- Can also do it by manipulating the phase
    - better because it does not lose energy

Fraunhoffer Diffraction

- Go Back to our original equation

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r+r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$

$$= \frac{-ikU_0}{2\pi z z'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

- where PS is the source-observation point distance
- $x_m = (zx' + z'x)/(z+z')$ $\qquad$ $x_m \to x$ as $z' \to \infty$
- $y_m = (zy' + z'y)/(z+z')$ $\qquad$ $y_m \to y$ as $z' \to \infty$
- $z_a = (zz')/(z+z')$ $\qquad$ $z_a \to z$ as $z' \to \infty$
- Now assume that x_0^2, y_0^2 are negligible

Fraunhoffer Diffraction

$$U_P = \frac{-ikU_0}{2\pi z z^{/}} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

$$= \frac{-ikU_0}{2\pi z z^{/}} e^{ik|PS|} \exp\left(\frac{ik}{2z_a} [x_m^2 + y_m^2] \right)$$

$$. \int_A \exp\left(\frac{-ik}{z_a} \left[(x_0 x_m + y_0 y_m \right] \right) dx_0 dy_0$$

- This is effectively saying that BOTH the source and the observation point are “far” from the aperture

Fraunhoffer Diffraction

- If we now generalise this by replacing the plane wave from the source (U_0/z') by a general distribution across the aperture $U(x_0,y_0)$
- Recognise that in a paraxial approximation a long way from the aperture it is the ANGLES that matter, not the positions
  - $u = kx_m/z_a$, $v = ky_m/z_a$

$$U_P = \frac{-ikU_0}{2\pi z z'} e^{ik|PS|} \exp\left(\frac{ik}{2z_a} [x_m^2 + y_m^2] \right)$$

$$. \quad \int_A \exp\left(\frac{ik}{z_a} \left[(x_0 x_m + y_0 y_m \right] \right) dx_0 dy_0$$

$$= B' \int_A U_0(x_0,y_0) \exp(-i [ux_0 + vy_0]) \, dx_0 dy_0$$

- In case you don't recognise it - the final expression is a 2-

D Fourier Transform relation!

Fraunhoffer Diffraction

$$U_P = \frac{-ik}{2\pi z} e^{ik|PS|} \exp\left(\frac{iz_a}{2k} [u^2 + v^2] \right)$$

$$\cdot \int_A U_0(x_0, y_0) \exp(-i [ux_0 + vy_0]) \, dx_0 dy_0$$

- relates x_0, y_0 space to u, v space
- e.g. for a square aperture and a plane wave -

$$U_P = \frac{-ik}{2\pi z} \exp\left(\frac{iz}{2k} [u^2 + v^2] \right) U_a \ell_x \ell_y sinc\left(\frac{\ell_x u}{2} \right) sinc\left(\frac{\ell_y v}{2} \right)$$

- the intensity is the square of this function
- For a circular function it is almost the same except that it's a Bessel function, not a sinc function

Fraunhoffer Diffraction

$$U_P = \frac{-ik}{2\pi z} e^{ik|PS|} \exp\left(\frac{iz_a}{2k} [u^2 + v^2] \right)$$

$$. \quad \int_A U_0(x_0,y_0) \exp(-i[ux_0 + vy_0]) \, dx_0 dy_0$$

- If this is so then we have a powerful technique
  - Applies for large distances
  - Applies for paraxial approximation
  - Maybe relax these conditions with more thought...
  - BUT any $U_0(x_0,y_0)$ can be used
  - We have computers - Have FFT - will compute!

Fourier Transforms for Fun and Profit

- Six functions you need to know

FUNCTION	FOURIER TRANSFORM
rect(x)	sinc(u/2)
ϐ(x)	1
comb(x)	comb(u/(2π))
Gaus(x)	Gaus(u/(2π))
step(x)	(1/2)ϐ(u/(2π)) + 1/(iu)
cyl(r)	J_1(u‘)/u‘

Fourier Transforms for Fun and Profit

- Fourier Transforms are linear
  - $\mathscr{F}(\alpha g + \beta h) = \alpha\mathscr{F}(g) + \beta\mathscr{F}(h)$
- Similarity (stretch one axis - contract the other)
  - If $\mathscr{F}(g(x,y)) = G(u,v)$
  - then $\mathscr{F}(g(\alpha x,\beta y)) = 1/|\alpha\beta| G(u/\alpha, v/\beta)$
- Shift property
  - If $\mathscr{F}(g(x,y)) = G(u,v)$
  - then $\mathscr{F}(g(x-\alpha,y-\beta)) = G(u,v)\exp(-i(\alpha u+\beta v))$
  - that's only a phase factor
- Parseval's Theorem
  - If $\mathscr{F}(g(x,y)) = G(u,v)$

$$\iint_{-\infty}^{+\infty} |g(x,y)|^2 \; dxdy = \iint_{-\infty}^{+\infty} |G(u,v)|^2 \; dudv$$

Fourier Transforms for Fun and Profit

- Convolution
  - If $\mathscr{F}$(g(x,y)) = G(u,v) and If $\mathscr{F}$(h(x,y)) = H(u,v)
  - then If $\mathscr{F}$(g*h) = G(u,v)H(u,v)
- Convolution in one space is multiplication in the other
  - If a(x) , A(u) and b(x), B(u) are transform pairs

$$\int_{-\infty}^{+\infty} a(\alpha)\, b(x-\alpha)\, d\alpha \;=\; a(x) * b(x) \;=\; \mathscr{F}^{-1}(A(u)\, B(u))$$

- convolution is...
  - commutative
  - distributive
  - shift invariant (change of origin)
  - associative
  - the δ function is the identity function for convolution

Approximation Criteria

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$

Assumptions made by this function derivation are that:

- We can approximate diffraction behaviour with a scalar potential.
- U and ∇U contribute negligible amounts to the integral except at the aperture.
- The values of U and ∇U at the aperture are the same as they would be in the absence of the aperture (no edge effects).
- Solutions satisfy the scalar wave equation.
- Solutions satisfy conditions of continuity and integrability [Green's Function].

Fresnel Approximations

$$U_P = \frac{-ikU_0}{2\pi z z'} \, e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

- Fresnel approximation applies when distances from the source and the observation points to the aperture are such that we can approximate spherical waves by surfaces of constant phase which are quadratic in their transverse variables.
- For an observation point (x,y,z) and aperture point (x_0,y_0,z_0) this implies:

$$z^3 >> \frac{\pi}{4\lambda}\left(\left(x - x_0\right)^2 + \left(y - y_0\right)^2\right)^2 .$$

- For the observation point (0,0,z), and aperture width Δx we require that:

$$\frac{z}{\Delta x} >> \left(\frac{\Delta x}{\lambda}\right)^{1/3}$$

Fraunhofer Approximations

$$U_P = \frac{-ik}{2\pi z} e^{ik|PS|} \exp\left(\frac{iz_a}{2k} [u^2 + v^2] \right)$$

$$. \quad \int_A U_0(x_0, y_0) \exp(-i[ux_0 + vy_0])\, dx_0 dy_0$$

- The Fraunhofer approximation applies when the incident and refracted waves may be approximated by plane waves.
- For the observation point (x,y,z) and aperture point (x_0, y_0, z_0) we require that

$$z_a >> \frac{\pi\left(x_0^2 - y_0^2\right)}{\lambda} .$$

- where $z_a \equiv (zz')/(z+z')$ and $z_a \rightarrow z$ as source distance $z' \rightarrow \infty$.
- Also known as the far field approximation.
- Tend to convert Cartesian observation coordinates (x,y,z) to angles (u,v) as they are a more “natural” coordinate set in a far field approximation.

Paraxial Approximation

- An additional approximation is required to remove the obliquity factor.
- We can remove the obiliquity factor if we can approximate cos(n,r) by unity.
- This approximation is valid if the vector r makes an angle less than ~30° to the normal.
- Required for both source and observation point.
- Equivalent to requiring that the source and obsevation points are close to the z-axis (hence the name).