Lecture 17

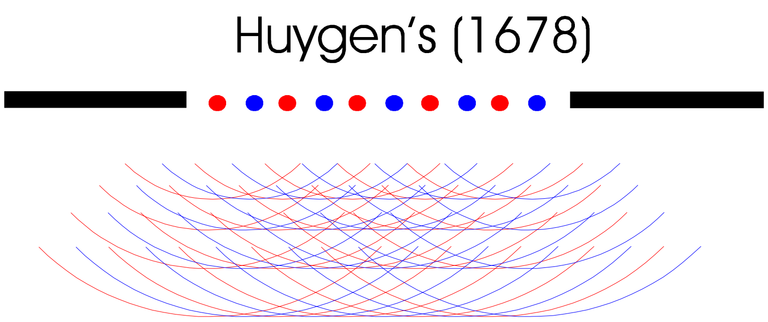
Diffraction

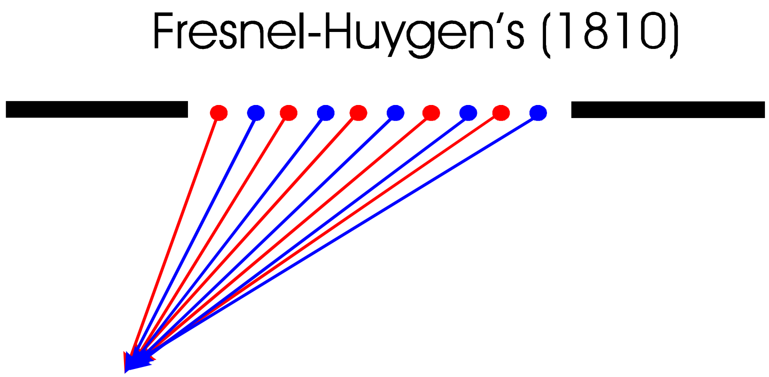
Why Does the Beam Spread?

- Interference in another guise?
- Because the wave equation says it has to
- Because the boundary of the wave cannot be sharp
  - Infinite E, H field gradients - bad
- Uncertainty principle
- Because it does - and we'd better find the math to explain it!!

Why Does the Beam Spread?

- Huygen’s Principle (1678)
  - Each point on the wavefront is a secondary radiator





  - Wave is the envelope of the secondary radiators
- Huygen’s-Fresnel Principle (1810)
  - Above is OK, but also need to account for phases
- These pre-date Maxwell and are for scalar waves
- No exact solution involving e-m theory of light until Sommerfeld in 1898

The Proof

- The proof has a number of stages
  - Simplify by considering scalar waves
  - Green's theorem applied to scalar waves
  - Relate value of field at point inside surface to surface values
  - More simplifications
  - Choose a tractable case
  - Consider the general case
  - Some interesting consequences (Babinet's principle)

Green’s Theorem Applied to Scalar Waves

$$\iint_S (W\nabla U - U\nabla W).dS = \iiint_V (W\nabla^2 U - U\nabla^2 W)\, dV$$

- Consequence of the divergence theorem
- f U and W satisfy the scalar wave equation

$$\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = -\frac{1}{v^2}\omega^2 U$$

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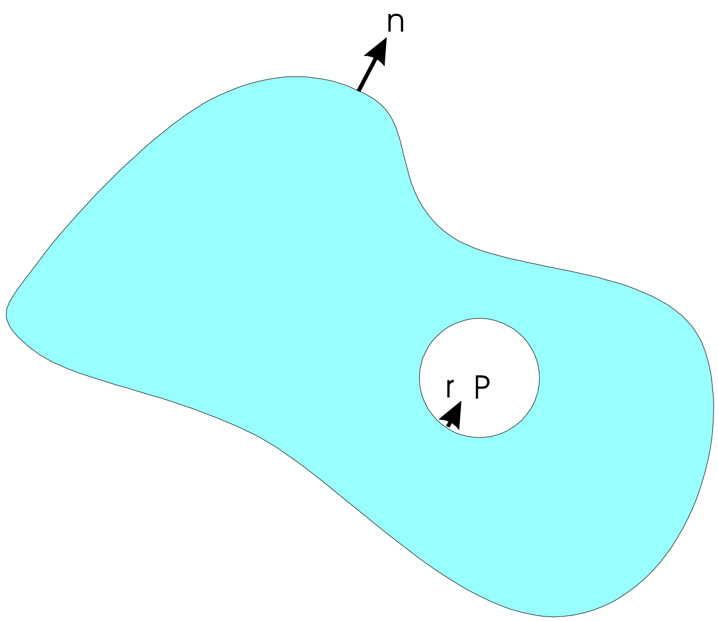
- where the last is true only for time dependence exp(-iωt)
- In that case RHS = 0 = LHS

$$\iint_S (W\nabla U - U\nabla W).dS = 0$$

Relate Point Inside Surface to Surface



- Let W be a spherical wave converging on origin P which is inside the surface (assume all time dependence as exp(-iωt))

$$W = W_0 \frac{1}{r} e^{ikr}$$

- Integrate over a closed surface
  - exclude the origin because W is infinite there
  - use a little sphere of radius ε to isolate origin

Relate Point Inside Surface to Surface

- consider integration over little sphere first

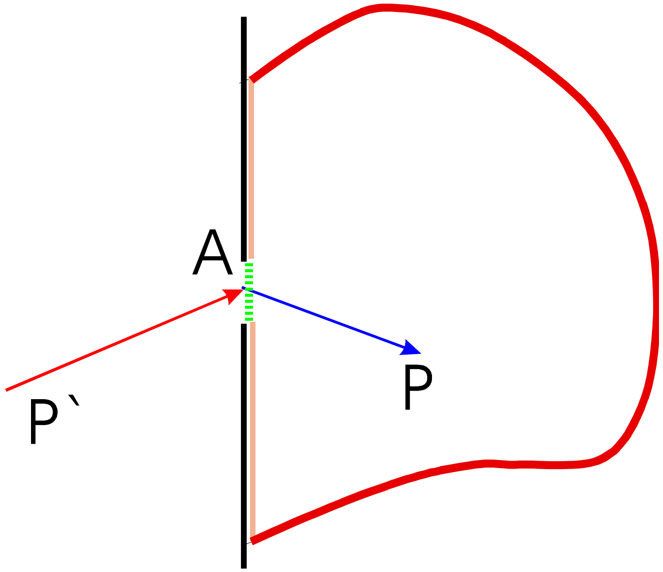
$$\iint_{S_\varepsilon} \left(W(\varepsilon) \left.\frac{\delta U}{\delta r}\right|_\varepsilon - U \left.\frac{\delta W}{\delta r}\right|_\varepsilon \right) \varepsilon^2 d\Omega = 4\pi U_P$$

- First term goes because U is locally smooth
- Second term simplifies as r,ε -> 0
- So if we now consider the full equation...

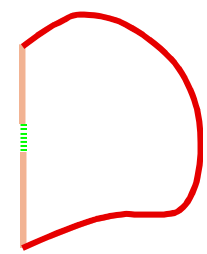
$$U_P = -\frac{1}{4\pi} \iint \left(U \frac{\delta}{\delta n} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \frac{\delta U}{\delta n} \right) dS$$

- This equation relates value at **origin** to value on surface
- BUT origins can be moved so it represents any point
- ∂/∂n is differential wrt normal to surface

More Simplifications/Tractable Case



- Consider a screen with a hole
- Integration surface is screen + hole + surface encompassing point of interest, P
- Assume U, ∇U are non-zero only at hole
  - Have same values as if screen wasn't there
- Assume U, ∇U are 0 elsewhere on surface
- So surface integral is only across hole
- U is a spherical wavefront from far side of hole
- $$U = U_0 \frac{1}{r^{/}} e^{ikr^{/}}$$



Now write down the full equation

Tractable Case?

$$U_P = \frac{U_0}{4\pi} \int_A \left(\frac{e^{ikr}}{r} \frac{\delta}{\delta n} \frac{e^{ikr'}}{r'} - \frac{e^{ikr'}}{r'} \frac{\delta}{\delta n} \frac{e^{ikr}}{r} \right) dS$$

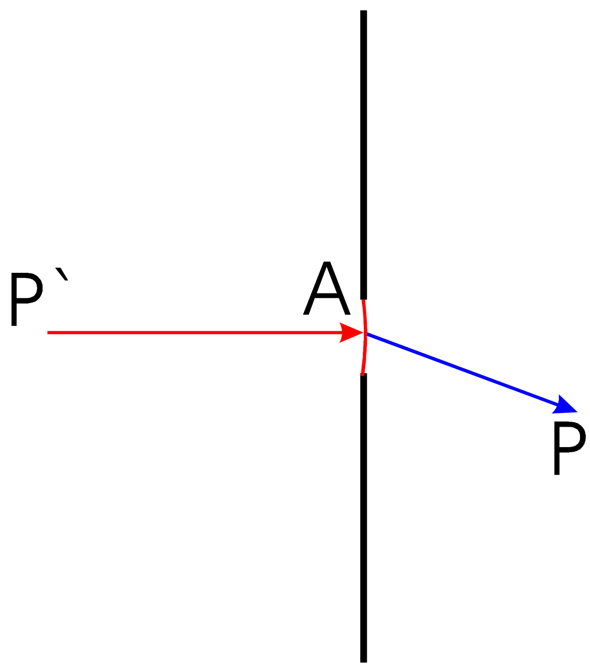
- Take a term...

$$\frac{\delta}{\delta n} \frac{e^{ikr}}{r} = \cos(n,r) \frac{\delta}{\delta r} \frac{e^{ikr}}{r} = \cos(n,r) \left(\frac{ikre^{ikr}}{r^2} - \frac{e^{ikr}}{r^2} \right)$$

- The first term dominates if kr >> 1

Tractable Case?

$$U_P = \frac{-ikU_0}{4\pi}\int_A \frac{e^{ik(r+r^{/})}}{rr^{/}}\left(\cos(n,r) - \cos(n,r^{/})\right)\, dS$$



- This is the Fresnel-Kirchoff integral formula (what all the fuss was about!)
- Needs to be applied to a specific case.

- Take a circular hole in a plate illuminated by a symmetrical source
- Take the surface to be the “spherical cap” equidistant from the source spanning the hole
- cos(n,r`) = -1 and r` is a constant

Tractable Case?

$$U_P = \frac{-ik}{4\pi} \frac{U_0 \, e^{ikr'}}{r'} \int_A \frac{e^{ikr}}{r} \left(\cos(n,r) + 1 \right) dS$$

- Can see the second spherical wave in the integrand
- Plus an "obliquity factor" [Defined as 0.5(cos(n,r)-cos(n,r'))]
- Also notice a phase shift of π/2 [NOT in Huygen's]
- Backwards propagation?

Babinet's Principle

$$U_P = \frac{-ik}{4\pi} \frac{U_0 e^{ikr'}}{r'} \int_A \frac{e^{ikr}}{r} \left(\cos(n,r) + 1 \right) dS$$

- If the hole is infinitely large then the effect must be as if the screen wasn't there

$$U_P = \frac{U_0 e^{ikr_{PP'}}}{r_{PP'}}$$

- Conversely if we have a small obscuration of the same size and shape as A

$$U_P = \frac{U_0 e^{ikr_{PP'}}}{r_{PP'}} - \frac{-ik}{4\pi} \frac{U_0 e^{ikr'}}{r'} \int_A \frac{e^{ikr}}{r} \left(\cos(n,r) + 1 \right) dS$$

- The obscuration is complementary to the aperture

Babinet's Principle

- A plane wave is represented as an infinite plane of secondary radiators
- This must be the sum of the two complementary screens
  - Infinite screen with a hole
  - obscuration same size and shape as hole