Lecture 11
Conducting Interfaces and Rough Surfaces

The Story So Far...

- The incident wave (Lecture 9)
  \[ \mathbf{k'} = (k'_x, k'_y, k'_z) = (k\sin\theta_1, 0, k\cos\theta_1) \]
- At the interface
  - Require spatial and temporal continuity
  - \[ \mathbf{k'} \cdot \mathbf{r} - \omega t = \mathbf{k} \cdot \mathbf{r} - \omega t = k \cdot \mathbf{r} - \omega t \]
  - no y components
  - no z component at interface (z=0)
  - \[ k'_x = k_x = k'_x \]
  - All the x components are equal
- We get =>
  - \[ \sin\theta_i = \sin\theta_1 \] - Law of reflection
  - \[ n_1 \sin\theta_1 = n_2 \sin\theta_2 \] - Snell's Law (n \sin\theta is conserved)

The Story So Far (II)...

- Boundary conditions for fields at the interface (Lecture 10)
- Tangential E and H fields are continuous
  - \[ (\mathbf{E}_1 + \mathbf{E}_2) \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n} \]
  - \[ (\mathbf{H}_1 + \mathbf{H}_2) \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n} \]
  - \[ (k'_x \mathbf{E}_1 + k \mathbf{E}_2) \times \mathbf{n} + (k'_x \mathbf{E}_1 + k \mathbf{E}_2) \times \mathbf{n} = (k_2 + k_1) \times \mathbf{E} \times \mathbf{n} \]
- We resolved components of \( \mathbf{E} \) or \( \mathbf{H} \) fields for \( k'_1 \) and \( k'_2 \).
- But, we noticed that there was a critical angle \[ \sin\theta_c = n_2/n_1 \]
- where there was no spatially varying z component of \( k'_2 \).
Evanescent TE Waves

- What's $\cos \theta_2$?
- Snell's law: $\sin \theta_2 = (n_1/n_2) \sin \theta_1 = \sin \theta_1 / \sin \theta_c$
- $\cos^2 \theta_2 = (1 - (n_1/n_2)^2 \sin^2 \theta_1) = (1 - \sin^2 \theta_1 / \sin^2 \theta_c)$
  - $\cos \theta_2$ is imaginary
  - Let $\beta^2 = 2 k^2 \sin^2 \theta_1 / \sin^2 \theta_c - 1 = -2 k^2 \cos^2 \theta_2$
- Transmitted wave propagates spatially as $\exp(i \cdot k \cdot r)$
- $\exp( i (x \cdot k \sin \theta_2 + z \cdot 2 k \cos \theta_2))$
- $\exp(-\beta z) \exp( i (x \cdot k \sin \theta_1))$
- Wave decays on space scale $1/\beta$
- for glass $n_2 = 1.5$ interface with air $n_1 = 1$
  - $\theta_\text{c} = \sin^{-1}(2/3) = 41.8^\circ$
  - at $45^\circ$, $1/\beta$ about $\lambda_0/2$
- Evanescent wave is the wave that penetrates the second medium decaying as $1/\beta$

Evanescent TE Waves - Phase Changes

- Look at boundary conditions
  - $r^* E_0 + r E_0 = 2 E_0$
  - $r^* E_0 \cdot k \cos \theta_1 - r E_0 \cdot k \cos \theta_1 = 2 E_0 \cdot 2 k \cos \theta_2$
  - $\beta = i \cdot k \cos \theta_2, \quad \alpha = -1 \cdot k \cos \theta_1$
- Solve for $r^* E_0$,$ r E_0$
  - $2 E_0 \cdot r^*/r E_0 = 2 \beta/(\alpha \cdot i \beta)$
  - $r^* E_0 / r E_0 = (\beta + i q)/(\alpha \cdot i \beta)$
- There are phase changes in the reflected beam
- Look at R, T
  - $T = 0$ because there is no z component in transmitted wave
  - $R = |r^* E_0 / r E_0|^2 = |(r^* E_0 / r E_0)(r^* E_0 / r E_0)^*| = 1$
- Total internal reflection as all beam energy is reflected
  - Occurs when $\theta_1 > \theta_\text{c}$

Reflection at a Conducting Interface

- Equations get complex - need approximations
- Normal Incidence - gets rid of angle effects
- $\mu_1 = \mu_2 = \mu_\text{0}$ (what about ferromagnetics?)
- Large conductivity $\sigma$ (not bad for metals)

Reflection at a Conducting Interface

- Boundary Conditions
  - $r^* E_0 + r E_0 = 2 E_0$
  - $r^* E_0 (i \cdot k / \mu_1) \cos \theta_1 - r E_0 (i \cdot k / \mu_1) \cos \theta_1 = 2 E_0 (2 k / \mu_2) \cos \theta_2$
- Not yet assuming that $\mu_1 = \mu_2$
- Now $2 k$ is complex
  - $2 k^2 = \omega^2 (\mu \epsilon) = \omega^2 (\mu_2 \epsilon_2) (1 + i \sigma_2 / (\epsilon_\omega))$
- Can get the angles from
  - $\sin \theta_2 = n_1 \sin \theta_1 / (n_2 + i k_0)$
  - But it makes the solution messy
- Assume normal incidence!
Reflection at a Conducting Interface

- Boundary Conditions
  - \( i^+ E_0 + i^+ E_0 = 2 E_0 \)
  - \( i^+ E_0 \left( i^+ k / \mu_1 \right) - i^+ E_0 \left( i^+ k / \mu_1 \right) = 2 E_0 \left( 2k / \mu_2 \right) \)
  - \( 2k^2 = \omega^2 \left( \mu_2 e_2 \right) \left( 1 + io_2 / (e_2 \omega) \right) \)
  - Still very complicated!!
  - In metals at low frequencies contributions of bound electrons negligible compared with conducting electrons.
  - So assume \( \sqrt{e_2} = n + ik = \sqrt{(io_2 / \omega \epsilon_0)} \)
  - \( \sqrt{i} = (1 + i)/\sqrt{2} \)
  - \( 2k = (1 + i) \sqrt{(\sigma_2 \mu_2 \omega/2)} \)
  - Soluble (sort of!!)

- The solution is left as an exercise...
- We can solve for the reflected component \( (i^- E_0 / i^+ E_0) \)
- \( R = \left| \frac{i^- E_0 / i^+ E_0}{(i^- E_0 / i^+ E_0)^*} \right| = 1 - 2 \sqrt{(2 \epsilon_1 \omega / \sigma_2)} \)
  - The higher the conductivity the higher is \( R \)
  - The lower the frequency the higher is \( R \)
  - Good conductors have little or no z-component to the reflected beam.
  - \( T = 2 \sqrt{(2 \epsilon_1 \omega / \sigma_2)} \) energy is dissipated in metal as Joule heating.
  - Assumed that the conductivity is the DC value.

A Simple Example

- We can extend our earlier analysis for a simple case of light incident on a metal in a vacuum. \( n_1 = 1, n_2 = n + ik \)
- The Reflectance \( R \) becomes
- \( R = \left| \frac{i^- E_0 / i^+ E_0}{(i^- E_0 / i^+ E_0)^*} \right| = ((n - 1)^2 + k^2)/((n + 1)^2 + k^2) \)
- \( k = 0 \Rightarrow \) dielectric case back
- \( k >> n \Rightarrow \) Reflected wave \( \sim 1 \)
- Sodium  \( \lambda = 589.3 \text{nm}, n=0.04, \quad \kappa=2.4 \quad T=0.1 \)
- Bulk Tin  \( \lambda = 589.3 \text{nm}, n=1.5, \quad \kappa=5.3 \quad T=0.2 \)
- Gallium  \( \lambda = 589.3 \text{nm}, n=3.7, \quad \kappa=5.4 \quad T=0.3 \)

Reflection from Non-Flat Surfaces

- All surfaces can be considered to be superposition of sinusoidal surfaces (Fourier analysis).
- A surface height function \( h(x) \) can be considered as Fourier components (given some conditions).
- The general series is of the form
  - \( z = h(x) = (1 / 2\pi) g(q) \exp(iqx) dq \)
- If \( h(x) \) is periodic
  - \( z = h(x) = \sum G(n) \exp(i n \pi x / \Lambda) \) for \( n = \{-\infty, ..., \infty\} \)
- Consider one such surface for which \( G(n) \neq 0 \) for only 2 components.
**The Sinusoidal Surface**

- Sinusoidal surface of perfect conductor ($T = 0$)
- Not strictly rigorous, but a limiting case
- Sum of incident and reflected fields = 0
  - $\sum E^0 \exp(i(k_x x + k_z z)) + \sum E^0 (x,z) = 0$
  - on the surface $z = h(x) = h_0 \cos(2\pi x/\Lambda)$
  - $h(x) = h_0/2 (\exp(2\pi x/\Lambda) + \exp(-2\pi x/\Lambda))$
- Solves at the boundary at $z = h(x)$ to
  - $\sum E^0 (x) = -\sum E^0 \exp(ik_x x)(1 + ik_z h_0 \cos(2\pi x/\Lambda))$

**Reflection from Non-Flat Surfaces**

- Look at simple solution for $k_z h_0 << 1$ ($\exp(x) = 1 + x$)
  - $\sum E^0 (x) = -\sum E^0 \exp(ik_x x)(1 + ik_z h_0 \cos(2\pi x/\Lambda))$
- Get the z component by conserving energy under total reflection
- Rewrite cos as sum of two complex exponentials
- Get three waves
  - $k_z, k_x + 2\pi/\Lambda, k_x - 2\pi/\Lambda$
  - Waves are at different angles
  - Diffraction grating
    - If we relax $k_z h_0 << 1$, get $k_x + 2mn/\Lambda$
- Rough surface can be simulated by summing sinusoids

**Reflection from Non-Flat Surfaces**

- Sinusoidal surface scatters at particular angles in steps of $2\pi/\Lambda$

**“Real World” Reflection**

- Examples of “typical” surface scattering by different types of surface (lobes represent polar diagrams of the scattered power)