Lecture 2

Linearity, Isotropy and Wave Solutions

Maxwell's Equations

- Div[ϵ_0 **E**] = - Div[**P**] + ρ
- Div[μ_0 **H**] = - Div[μ_0 **M**]
- Curl[**E**] = - $\partial/\partial t$ (μ_0 **H**) - $\partial/\partial t$ (μ_0 **M**)
- Curl[**H**] = $\partial/\partial t$ (ϵ_0 **E**) + $\partial/\partial t$ (**P**) + **J**
- $\mu_0 = 4\pi \cdot 10^{-7}$ in SI units
- $\epsilon_0 = 10^{-9}/(36\pi)$ in SI units

In The Physicist's World

- Everything is linear!!
- (In the real world everything is non-linear with very few exceptions)
- Polarisation (**P**), Magnetisation (**M**) and current density (**J**) are effects, not causes
- Therefore **P** and **J** are "proportional" to **E**
- And **M** is "proportional" to **H**

Tensors!!!

- Don't worry they don't last long!!
- The electric susceptibility is in general a tensor
- **P** = ϵ_0 χ_e **E**

$$\begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix}$$

Symmetry and Isotropy

- If material has no preferred axis, then it said to be **isotropic**
- If it is isotropic then the off-diagonal elements of the matrix are zero and the diagonal elements are equal
- **P** = $\epsilon_0 \chi_e$ **E**

$$\begin{pmatrix} \chi & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{pmatrix}$$

Same For Magnetics

- **M** = χ_m **H**
- For isotropic media the magnetic susceptibility reduces from a tensor to a scalar
- **M** = χ_m **H**

Same For Current Density

- **J** = σ **E**
- For a linear, isotropic medium σ is a scalar

Recap

- For **linear** media the response is proportional to the stimulus and the constant of proportionality is a **tensor**
- For **linear**, **isotropic** media the response is proportional to the stimulus and the constant of proportionality is a **scalar**
- Almost all cases are **linear**
- A large number are **linear and isotropic**

MEs In a Linear, Isotropic Medium

- Div[ϵ_0 **E**] = - Div[$\chi_e \epsilon_0$ **E**] + ρ
- Div[μ_0 **H**] = - Div[$\chi_m \mu_0$ **H**]
- Curl[**E**] = - $\partial/\partial t$ (μ_0 **H**) - $\partial/\partial t$ ($\chi_m \mu_0$ **H**)
- Curl[**H**] = $\partial/\partial t$ (ϵ_0 **E**) + $\partial/\partial t$ ($\chi_e \epsilon_0$ **E**) + σ**E**
- This might have some solutions!!

MEs In a Linear, Isotropic Medium

- Div[**E**] = ρ/ϵ
- Div[**H**] = 0
- Curl[**E**] = - $\partial/\partial t$ (μ **H**)
- Curl[**H**] = $\partial/\partial t$ (ϵ **E**) + σ**E**
- $\epsilon = \epsilon_0 (1 + \chi_e)$
- $\mu = \mu_0 (1 + \chi_m)$

MEs In a Linear, Isotropic Medium

- Curl[Curl[**E**]] + $\partial^2/\partial t^2$ ($\mu\epsilon$**E**) + $\partial/\partial t$ ($\mu\sigma$**E**) = 0
- Curl[Curl[**H**]] + $\partial^2/\partial t^2$ ($\mu\epsilon$**H**) + $\partial/\partial t$ ($\mu\sigma$**H**) = 0
- It is well-known that (ie I found it in a book somewhere)
- Curl[Curl[**X**]] = Grad[Div[**X**]] - Del2[**X**]
- but...
- Div[**H**] = 0
- so....
- Del2[**H**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**H**) - $\partial/\partial t$ ($\mu\sigma$**H**) = 0
- A WAVE EQUATION!!!
- and by similar means
- Del2[**E**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**E**) - $\partial/\partial t$ ($\mu\sigma$**E**) = Grad[ρ/ϵ]
- which if $\rho = 0$ (charge-free medium) is the same as the equation for **H**

MEs In a Linear, Isotropic Medium

- Del2[**H**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**H**) - $\partial/\partial t$ ($\mu\sigma$**H**) = 0
- Del2[**E**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**E**) - $\partial/\partial t$ ($\mu\sigma$**E**) = 0
- For a **linear**, **isotropic**, **charge-free** medium
- Doesn't have to be **charge-free**, but the derivation is easier that way!!
- It doesn't even have to be **isotropic**, but the derivation is easier that way!!
- Phase speed in vacuum (χ_m, χ_e = 0) is
- $(\mu_0 \epsilon_0)^{-1/2} = 3 \times 10^8 \text{ ms}^{-1} = c$

Dissipation in MEs

- For a **linear**, **isotropic**, **charge-free** medium
- Del2[**H**]] - $\partial^2/\partial t^2$ (με**H**) - ∂/∂t (μσ**H**) = 0
- Del2[**E**]] - $\partial^2/\partial t^2$ (με**E**) - ∂/∂t (μσ**E**) = 0
- The third term in the above equation represents the dissipation
- For a **linear**, **isotropic**, **charge-free**, **non-conducting** medium (σ = 0) the wave does not dissipate
- A Vacuum is a particular case of the above!!

Transverse Waves?!!

- Del2[**H**]] - $\partial^2/\partial t^2$ (με**H**) = 0
- Del2[**E**]] - $\partial^2/\partial t^2$ (με**E**) = 0
- The most general plane wave solutions of the above are
- **E** = **x** [f_+ (z - vt) + f_- (z + vt)]
- **H** = $(\epsilon/\mu)^{1/2}$ **y** [f_+ (z - vt) + f_- (z+vt)]
- Note that **E** and **H** are perpendicular to each other and to the direction of propagation - z
- But in a vacuum, these are transverse waves in what?
- And thereby hangs an aethereal tale.....

Monochromatic Waves

- **E** = E_0 **x** cos(**k** . **r** ± ωt + ϕ)
- ϕ is an arbitary phase
- ω is the angular frequency
- **k** defines both the direction of propagation (DOP) and the wave vector |**k**| = 2π/λ = k
- **k** . **r** ± ωt defines a surface of constant phase which is displaced in time at a speed ω/k which is called the "phase speed"

Velocity In a Medium

- If χ_e, χ_m are # 0, then the phase speed is reduced
- Phase speed becomes c/n
- n = $(\epsilon/\epsilon_0 \,.\, \mu/\mu_0)^{1/2}$
- n is the "refractive index" and is > 1 (mostly)
- If either of χ_e, χ_m are dependent upon frequency, then the medium has "dispersion"
- This means that if n is frequency-dependent, then the medium is dispersive

Complex Numbers

- Complex numbers for physicists are a “convenience” for solving problems
- Rely on the real-world fiction that $\sqrt{-1}$ can be represented by i (the engineers use j)
- A complex number can be written as $C = a + ib$
- The conjugate complex is $C^* = a - ib$
- The modulus is $|C| = \sqrt{(a^2 + b^2)}$
- The argument is $Arg(C) = \tan^{-1}(b/a)$

Complex Exponentials - Phasors

- If $C = \cos(a) + i \sin(a)$
- Then it is shown in all the best text books that you can write $C = \exp(ia)$
- In fact any complex number C can be written as $C = |C| \exp(i \arg(C))$
- Since manipulation of exponentials in integration and differentiation is very easy, it is tempting to try to manipulate any wave equation into this format

Complex Superposition

- If $E_0 \mathbf{x} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)$ is a solution
- Then so is $E_0 \mathbf{x} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)$
- And so is $i \cdot E_0 \mathbf{x} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)$
- By the principle of linear superposition, the sum of the above solutions is also a solution which can be expressed as
- $E_0 \mathbf{x} \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi)$
- or $\mathbf{x} \cdot E \cdot \exp(-i\omega t)$ where E is complex and contains all spatial and phase components $E = E_0 \exp i(\mathbf{k} \cdot \mathbf{r} + \phi)$

Differentiation wrt Time

- For any quantity $C \exp(-i\omega t)$ where C is independent of time
- $\partial/\partial t\, (C \exp(-i\omega t)) = -i\omega\, C \exp(-i\omega t)$
- Similarly $\mathrm{Int}[C \exp(-i\omega t)] = -1/i\omega\, C \exp(-i\omega t)$
- What does this do to Maxwell’s equations?
- $\mathrm{Div}[\epsilon_0 \boldsymbol{E}] = -\mathrm{Div}[\boldsymbol{P}] + \rho$ - unchanged
- $\mathrm{Div}[\mu_0 \boldsymbol{H}] = -\mathrm{Div}[\mu_0 \boldsymbol{M}]$ - unchanged
- $\mathrm{Curl}[\mathbf{E}] = -\partial/\partial t\, (\mu_0 \mathbf{H}) - \partial/\partial t\, (\mu_0 \mathbf{M})$ becomes
- $\mathrm{Curl}[\boldsymbol{E}] = i\omega\mu_0 \boldsymbol{H} + i\omega\mu_0 \boldsymbol{M}$
- $\mathrm{Curl}[\mathbf{H}] = \partial/\partial t\, (\epsilon_0 \mathbf{E}) + \partial/\partial t\, (\mathbf{P}) + \mathbf{J}$ becomes
- $\mathrm{Curl}[\boldsymbol{H}] = -i\omega\epsilon_0 \boldsymbol{E} - i\omega \boldsymbol{P} + \boldsymbol{J}$

ME Monochromatic Wave Eqn

- Del2[***H***]] + $\omega^2 \mu \epsilon$***H*** + $i\omega\mu\sigma$***H*** = 0
- Del2[***E***]] + $\omega^2 \mu \epsilon$***E*** + $i\omega\mu\sigma$***E*** = 0
- $\mu_0 = 4\pi \,.\, 10^{-7}$ in SI units
- $\epsilon_0 = 10^{-9}/(36\pi)$ in SI units
- If we now REDEFINE ϵ as being $\epsilon + i\sigma/\omega$, we can automatically incorporate conducting media in the solutions with a complex dielectric constant
- This implies that the effect of conductance is the the polarisation **P** is still proportional to the field **E**, but there is a phase lag between the field and the polarisation
- We can also look at the same arguments for the spatial differential Del2[] and find that in a charge-free medium Del2[***X***] = $-k^2$ ***X*** where k is the wave vector

Dispersion Relation

- Putting all these components into the wave equation, we find that
- $k^2 = \omega^2 \epsilon \mu$ for a linear, isotropic, charge-free medium
- Remember that ϵ is generally complex and therefore so is k
- Remember also that $\mu \approx \mu_0$ in most cases
- If we define refractive index to be complex following ϵ, then
- $k^2 = \omega^2 / c^2 \,.\, n^2$
- This is the “dispersion relationship” and measures how the wave vector k, varies with frequency

n and ϵ

- Most materials are non-magnetic, assume $\mu = \mu_0$
- $\epsilon = \epsilon_r + i\,\epsilon_i = n^2 \epsilon_0$
- So if $n = n_r + i\, n_i$
- Then $\epsilon_r / \epsilon_0 = n_r^2 - n_i^2$
- And $\epsilon_i / \epsilon_0 = 2\, n_r\, n_i$
- What’s the point?
- We can measure the refractive index very easily
- therefore we can relate the macroscopic n to the microscopic, complex ϵ

Bringing It Back to Reality

- The wave propagates like exp $i(\mathbf{k} \,.\, \mathbf{r} - \omega t + \phi)$
- Suppose it propagates along the z axis -> kz
- The wavelength in free space is λ_v
- and the wavevector is k_v
- Look at the spatial part only
- $\exp(ikz) = \exp\{i\,(n_r + i\,n_i)\,k_v\, z\}$
- $\exp(ikz) = \exp(i\, n_r\, k_v\, z) \quad \exp(-\, n_i\, k_v\, z)$
- First term is phase factor, second is a decay term
- Measure the phase speed - gives n_r (historically n)
- Measure the decay - gives n_i (historically κ)

Refractive Indexes

- Vacuum = 1.00 (surprise, surprise)
- Glasses = 1.57-1.77
- Sapphire = 1.77, Diamond = 2.42
- Silicon = 3.8 + 0.4i
- Silver = 2.3 + 3.8i
- Air = $1.0 + 290 \cdot 10^{-6} (\rho/\rho_{stp})$