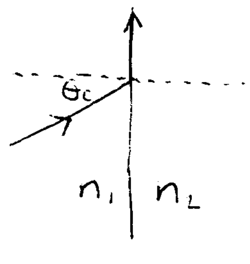
PHY 353S MIDTERM ANSWERS

1 The critical angle is the angle for which an angle of incidence greater than θ_c, the refracted wave does not exist.

From Snell's law : $n_1 \sin\theta_1 = n_2 \sin\theta_2$



so at the critical angle, we will have $\theta_2 = \frac{\pi}{2}$, $\theta_1 = \theta_c$

$\therefore n_1 \sin\theta_c = n_2$

and $\sin\theta_c = \frac{n_2}{n_1}$

Note that for θ_c to be real, $n_1 > n_2$ so that $\sin\theta_c < 1$
Thus, the conditions for total internal reflection are

$\theta_1 > \theta_c$ (incident angle > critical angle)

$n_1 > n_2$ (light is incident from denser medium)

2 Assume that we begin with RHC polarized light.
The Jones vector is $\vec{E}_i = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}$

Now suppose the $\lambda/4$ plate has its fast axis along the x-axis.
The Jones matrix for this element is $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} e^{i\pi/4}$

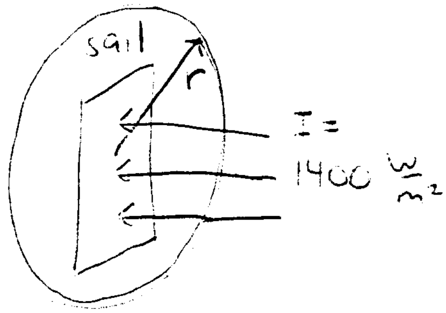
So the final state of the light after the $\frac{\lambda}{4}$ plate is given by

$$\vec{E}_t = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\pi/4}$$

This corresponds to linearly polarized light polarized at $+45°$ relative to the x-axis. (With a "global" 45° phase shift)

(This is one possible answer to the question. There are 4 possible answers, depending on your choice of $\vec{E}_i$ and the fast axis of the $\frac{\lambda}{4}$ plate)

3



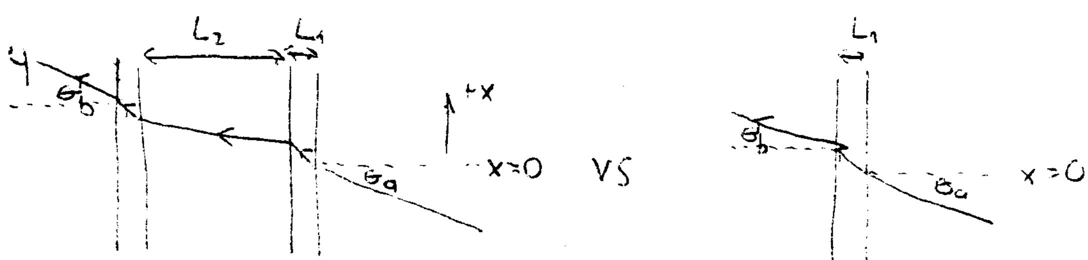
The momentum delivered by the sunlight is $p = 2\frac{E}{c}$
where E is the energy in the light beam.
Since the ship sails directly away from the sun I have dropped the vector sign on p, assuming that the ship travels in the same direction as the light beam.
Since the sail is perfectly reflecting the momentum transfer is $2 \cdot \frac{E}{c}$
Thus, $F = 2\frac{P}{c}$, where P = power in light beam

But, $P = IA$

$$\therefore F = \frac{2IA}{c}$$

So, $$A = \frac{cF}{2I} = \frac{3\times10^8 \frac{m}{s} \cdot 1N \left[W \frac{s}{m}\right]}{2\times1.4\cdot10^3 \frac{W}{m^2}} = 1.1\times10^5 m^2$$

Hence, the radius of the sail is $r = \sqrt{\frac{1.1\times10^5}{\pi}}\ m = 187 m$



As much as the thought of applying Snell's law 6 times appeals to me (and trudging through all the high school geometry) I'd much rather use matrix methods.

The initial ray vector is $\begin{bmatrix} a \\ b \end{bmatrix} = R_i$, with a = distance from x-axis
$b = \tan\theta_a$

WLOG, let $a = 0$

also let $n = 1$ (air on each side of the window)

For the case with just the pane of glass.

$$\vec{R}_F = \begin{pmatrix} 1 & 0 \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/n_1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$= \begin{pmatrix} bL_1/n_1 \\ b \end{pmatrix}$$

For the case with the unknown gas

$$\vec{R}_F = \begin{pmatrix} 1 & 0 \\ 0 & n_1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & n_g/n_1 \end{pmatrix} \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_g \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/n_1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$= \begin{pmatrix} b\left(\frac{2L_1}{n_1} - \frac{L_2}{n_g}\right) \\ b \end{pmatrix}$$

The directional element of $\vec{R}_F$ in each case is the same ($=b$, i.e. the incident direction is the same as the emergent direction)

The other element is the deflection of the beam from the optic axis. of course, we expect these to be different.

5 The intensity at the point of recombination is

$$I_F = I_1 + I_2 + 2\sqrt{I_1 I_2}\, \mathrm{Re}[\gamma_{12}(\tau)]$$

$$\gamma_{12}(\tau) = \exp(-i\omega\tau) \exp\left(\frac{-|\tau|}{\tau_0}\right)$$

$$\mathrm{Re}[\gamma_{12}(\tau)] = \cos\omega\tau \exp\left(\frac{-|\tau|}{\tau_0}\right)$$

We know $\tau_0 = 1 \times 10^{-11}$ s

and $\tau = \frac{0.5 \times 10^{-2} m}{3 \cdot 10^{8} \frac{m}{s}} = 1.67 \times 10^{-14} s$

Also, the beam is split into two at a 50/50 beam splitter. Thus, if $I_i = I$, then $I_1 = I_2 = \frac{I}{2}$

and $I_f = \frac{I}{2} + \frac{I}{2} + 2\sqrt{\frac{I}{2} \cdot \frac{I}{2}} \cos \omega \tau \exp\left(-\frac{|\tau|}{\tau_0}\right)$

$= I + I \cos \omega \tau \exp\left(-\frac{|\tau|}{\tau_0}\right)$

Now, $\omega = 2\pi\nu = 2\pi \frac{c}{\lambda} = 3.43 \cdot 10^{15} s^{-1}$

So,

$$I_f = I + I(-0.94) \exp\left(-\frac{1.67 \cdot 10^{-11}}{1 \cdot 10^{-11}}\right)$$
$$= I + I(-0.94)(0.188)$$
$$= 0.82 I$$

Note, different answers may have been found depending on how you rounded the numbers.