The Torsion Pendulum
(One or two weights)

Exercises I through V form the one-weight experiment.
Exercises VI and VII, completed after Exercises I -V, add one weight more.

Preparatory Questions:

1. The torsional stiffness, or the torsion constant, κ, is defined as the amount of torque needed to rotate one end of a wire by 1 radian about the longitudinal axis of the wire with the other end being fixed. What do you think will happen to the torsion constant as you use a wire of the same material and length but with a larger diameter?

2. For oscillation of mass m on spring, the angular frequency ω_0 is introduced as $\omega_0^2 = k/m$, where k is the spring constant. Using the resemblance of linear and angular quantities, derive a similar equation for the angular frequency of torsional oscillations in absence of damping.

3. Find in your textbook [1] or [2] and write into the lab notebook the moment of inertia of the following objects about their axes of symmetry through the center of mass:
   (a) A point mass M rotating in a circle of radius R
   (b) A solid disk or a cylinder of mass M and radius R about an axis of the highest order of symmetry and through the center of mass
   (c) A hollow cylinder of mass M and R_{in} as the inner radius of the cylinder and R_{out} as its outer radius
   (d) A thin rod of length L and mass M about an axis through the center of mass and perpendicular to the rod

   You will use the selected formulae in your experiment.

1. Introduction

Torsion is a type of stress, which is easier to explain for a uniform wire or a rod when one end of the wire is fixed, and the other end is twisted about the axis of the wire by an external force. The external force causes deformation of the wire and appearance of counterforce in the material. If this end is released, the internal torsion force acts to restore the initial shape and size of the wire. This behavior is similar to the one of the released end of a linear spring with a mass attached.

Attaching a mass to the twisting end of the wire, one can produce a torsion pendulum with circular oscillation of the mass in the plane perpendicular to the axis of the wire.

To derive equations of rotational motion of the torsion pendulum, it would be useful to recall a resemblance of quantities in linear and rotational motion. We know that if initially a mass is motionless, its linear motion is caused by force F; correspondingly, if an extended body does not rotate initially, its rotation is caused by torque τ. The measure of inertia in linear motion is mass, m, while the measure of inertia in rotational motion is the moment of inertia about an axis of rotation, I. For linear and angular displacement in a one-dimensional problem, we use either x or θ. Thus, the two equations of motion are:

$$F_x = ma_x \quad \text{and} \quad \tau = I\alpha \qquad (1)$$

where a_x and α are the linear and the angular acceleration.
If the linear motion is caused by elastic, or spring, force, the Hooke's law gives $F_x = -kx$, where k is the spring constant. If the rotation is caused by torsion, the Hooke's law must result in

$$\tau = -\kappa\theta \qquad (2)$$

where κ is the torsion constant, or torsional stiffness, that depends on properties of the wire. It is essentially a measure of the amount of torque required to rotate the free end of the wire 1 radian.

Your answer to the Preparatory Question 2 gives the following relationship between the moment of inertia I of an oscillating object and the period of oscillation T as:

$$I = \left(\frac{T}{2\pi}\right)^2 \kappa \qquad (3)$$

This relationship is true for oscillation where damping is negligible and can be ignored. Otherwise the relationship between I and κ is given by

$$I = \frac{\kappa}{\omega_0^2} \qquad (3^*)$$

where ω_0 can be found from $\omega = \sqrt{\omega_0^2 - \left(\frac{c}{2I}\right)^2} \qquad (3^{**})$

$\omega = \frac{2\pi}{T} = 2\pi f$; f is the frequency of damped oscillation; and c is the *damping coefficient*.
The relationship between the torsion constant κ and the diameter of the wire d is given in [3] (check your answer to the Preparatory Question 1) as

$$\kappa = \frac{\pi G d^4}{32 l} \qquad (4)$$

where l is the length of the wire and G is the shear modulus for the material of the wire.

As any mechanical motion, the torsional oscillation is damped by resistive force originating from excitation of thermal modes of oscillation of atoms inside the crystal lattice of the wire and air resistance to the motion of the oscillating object. We can estimate the torque of the resistive force as being directly proportional to the angular speed of the twisting wire, i.e. the torque $\tau_R = -c\,d\theta/dt$ (recall the drag force on mass on spring in viscose medium as $R = -bv$). Combining Eq.(1), (2) and the expression for τ_R, we obtain the equation of motion of a torsional pendulum as follows:

$$I\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+\kappa\theta=0 \qquad (5)$$

The solution of Eq.(5) is similar to the solution of the equation for damped oscillation of a mass on spring and is given by:

$$\theta = Ae^{-\alpha t}\cos(\omega t+\varphi) \qquad (6)$$

where $\alpha = c/2I$ (7)

and $\alpha = \beta^{1}$ with β being the time constant of the damped oscillation; c is the damping coefficient; ω is the angular frequency of torsional oscillation measured in the experiment; and φ can be made zero by releasing the object on the wire at a position of the greatest deviation from equilibrium.

Equation (6) can be used to calculate c (damping coefficient) and β (time constant = amount of time to decay e times) with DataStudio interface and software.

Another important formula is $\alpha = \omega_0 /2Q$, where Q is the *quality factor* and $\omega_0^2 = \kappa / I$ (see Eq.3'). The ratio

$$\zeta = \alpha/\omega_0 = (2Q)^{-1} \qquad (8)$$

is called the *damping ratio*.

2. Experiment

The experiment setup includes:

- Wires of 3 different diameters
- Rotary Motion Sensor (RMS) Connected to Science Workshop Interface
- Force Sensor
- Lower Clamp
- Upper Clamp
- Disk
- Hollow cylinder
- Rod with 2 point masses attached
- String (about 70 cm at least)
- Mass scale
- Micrometer
- Ruler

1. This experiment contains wires of 3 different thicknesses. Use the micrometer to measure the diameters of each wire.

FIG.: 1

http://www.upscale.utoronto.ca/PVB/Harrison/Micrometer/Micrometer.html

2. The support rod can be either mounted on a stand as in Fig.1, or clamped to the table as in Fig. 2. Make sure that the support rod it securely mounted and is quite stable.

3. Make sure that the rotary motion sensor (RMS) and the lower clamp are parallel.

4. Now slide the upper clamp onto the shaft of the RMS. Make sure that the guide mark on the upper clamp aligns with the shaft.

5. Take the wire with the smallest diameter and screw it to both the upper and the lower clamps. Adjust the distance between the rotary motion sensor and the lower clamp as needed to make sure that the wire is tight and fits perfectly into the grooves of the screws (Fig.2).

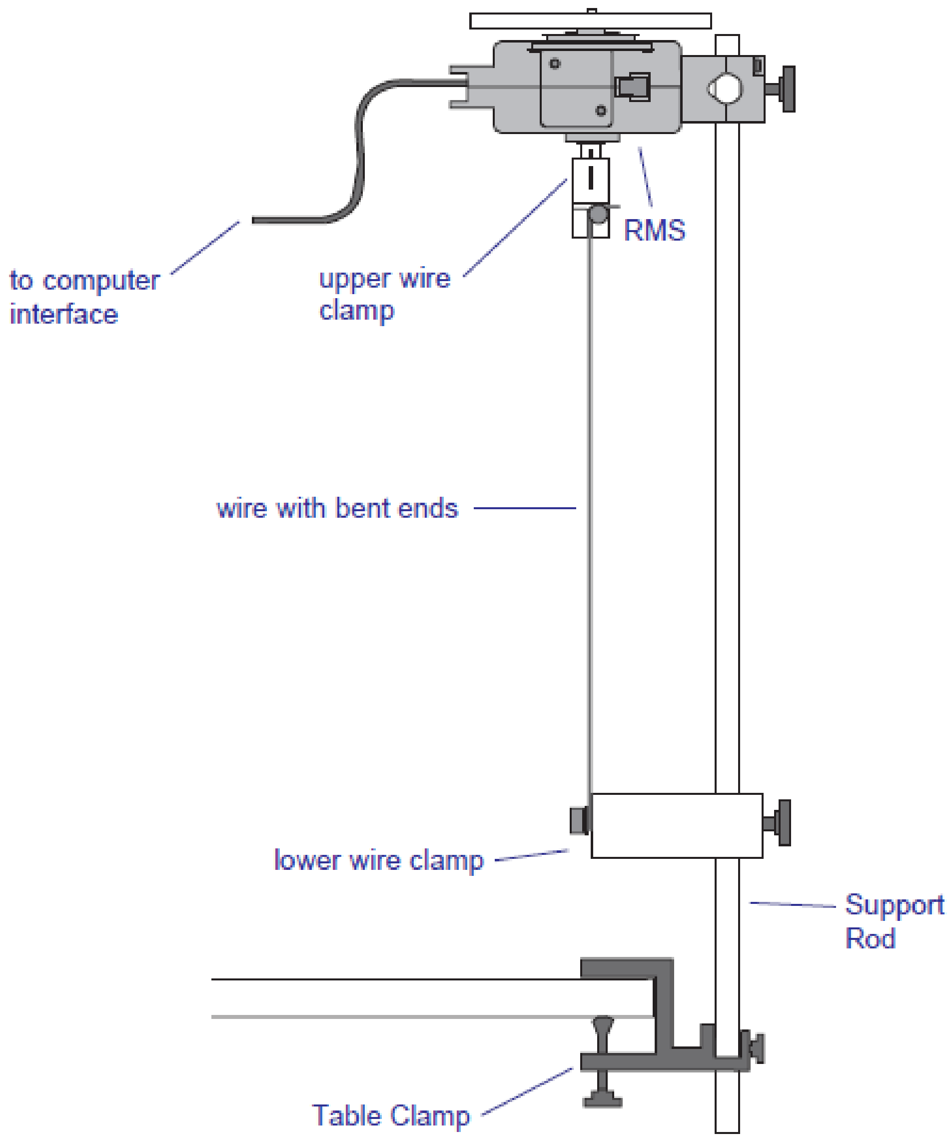

FIG.: 2

6. Measure and record the radius of the medium pulley of the RMS.
7. Take a string of about 50-cm length, tie it to the small pulley on the RMS and wrap around the medium pulley about 3 times. Remember to record how much you used. After wrapping around the medium pulley, leave about 20 cm of string before cutting of the rest. Make a loop with the string near the end where you just cut off the excess string. This loop will allow you to hook on the force sensor.

8. Connect the RMS cables to Channel 1 & 2 on the Science Workshop Interface and the force sensor to Channel A. Set a sample rate of 50 Hz and Med(10x) sampling option for the force sensor.

Obtain approximately 5 runs of data for each exercise.

Exercise 1: **Calculating the torsion constant κ**

Hook the force sensor onto the string loop while the string is slack. Tare the force sensor. Hold the force sensor parallel to the rotary motion sensor as in Fig. 3. Start the DataStudio timer and pull on the force sensor. After about 1 revolution, stop the Data Studio timer.

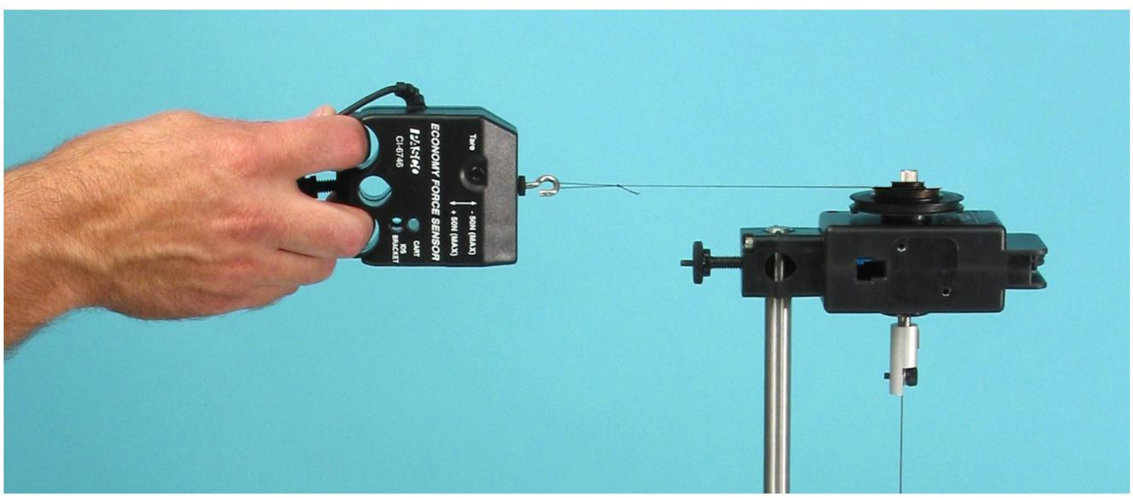

FIG.: 3

9. Open the DataStudio calculation option and create a function for torque. We are using the following formula:

$$\tau = r \times F$$

   where r is the radius of the pulley that was used (in this case the medium pulley) and F is the force exerted, which is recorded by the Force Sensor. Therefore, r is defined as an Experimental Constant and F is a Data Measurement on DataStudio.

10. Create a graph of Torque vs. Angle. The graph of Torque vs. Angle should be approximately linear. Perform a linear fit on the data and find the torsion constant κ for the wire.

Exercise 2: **Comparing the values of the moment of inertia of the disk obtained with different methods**

1. Remove the force sensor and untie the string from the setup from Exercise 1.

2. Make all necessary measurements (mass, geometrical parameters) to determine the moment of inertia of the disk about the axis through the center of mass I_{D1}.

3. Place the disk on top of the pulley and screw it in (Fig.4).

4. Open a New Experiment in DataStudio and plot a graph of Angle vs. Time. Use a sample rate of 200Hz.

5. Turn the disk about 90^{o} from its equilibrium position. Start the DataStudio timer and release the disk. Let the disk oscillate for about 5 – 7 seconds and then stop the DataStudio timer.

6. Obtain the period of oscillation through either a sine fit or by using the Smart Tool.

7. Using Eq.3, obtain the moment of inertia of the disk I_{D2}.

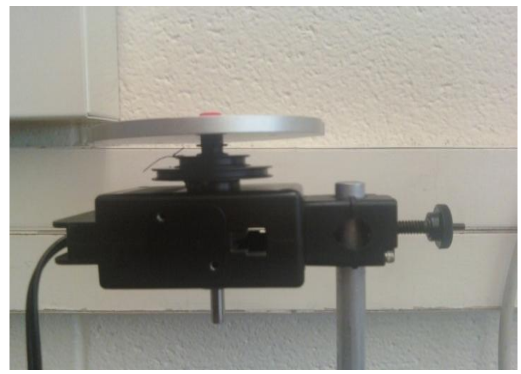

FIG.: 4

8. Compare the values of I_{D1} and I_{D2} by calculating the difference $\Delta I_D = | I_{D1} - I_{D2} |$. Obtain the value of ΔI_{D1} by using the formula for error propagation. Is the difference between two values of the moment of inertia of the disk within the error of measurements? Comment on your results.

Exercise 3: **Comparing the values of the moment of inertia of the system "a disk + a hollow cylinder" obtained with different methods**

FIG.: 5

1. Measure all necessary parameters of the hollow cylinder to calculate its moment of inertia I_{C1} about the axis through the center of mass.

2. Place the cylinder on top of the disk as in Fig. 5. Make sure the screws in the cylinder fit into the grooves on the disk.

3. Open a New Experiment in DataStudio and plot a graph of Angle vs. Time. Use a sample rate of 200Hz.

4. Rotate the disk with the cylinder about ¼ of the way from its equilibrium position. Start the DataStudio timer and release the disk. Let the disk oscillate for about 5-7seconds and then stop the DataStudio timer.

5. Obtain the period of oscillation through either a sine fit or by using the Smart Tool and calculate the moment of inertia I_{C2}.

6. Compare the values of I_{C1} and I_{C2} by calculating the difference $\Delta I_C = | I_{C1} - I_{C2} |$. Obtain the value of ΔI_{C1} by using the formula for error propagation. Is the difference between two values of the moment of inertia of the disk within the error of measurements? Comment on your results.

Exercise 4: **Comparing the values of the moment of inertia of the rod and two point masses obtained with different methods**

1. Remove the disk and the cylinder from the RMS.

2. Measure and record the mass and length of the rod and the mass of each point mass included. Clamp the 2 point masses on the 2 ends of the rod. Calculate the moment of inertia of the system I_{R1}. Using the formula for error propagation, calculate the uncertainty ΔI_{R1}. Except the uncertainty of direct measurements, what other factors influence inaccuracy of obtained value for I_{R1}?

3. Invert the 3-step pulley on the RMS. Superpose the center of the rod on the center of the round pulleys and screw it in as in Fig.6.

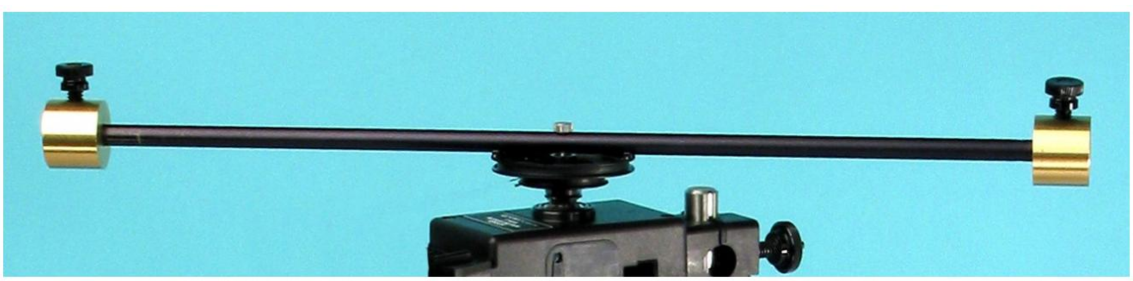

FIG.: 6

4. Open a New Experiment in DataStudio and plot a graph of Angle vs. Time. Use a sample rate of 200Hz.

5. Rotate the rod with the 2 point masses about ¼ of the way from its equilibrium position. Start the DataStudio timer and release the rod. Let the rod oscillate for about 5-7seconds and then stop the DataStudio timer.

6. Obtain the period of oscillation through either a sine fit or by using the Smart Tool. Using the relationship (3), calculate the moment of inertia I_{R2} of the system ”the rod + two point masses”.

7. Calculate the difference $\Delta I_R = | I_{R1} - I_{R2} |$ and compare it with ΔI_{R1}. Comment on your results.
8. Which exercise: 2, 3 or 4, gives more accurate result for the moment of inertia by measuring the mass and the geometrical parameters of the object compared to the method of measuring the period of oscillation? Why?

***Exercise 5:* Observing damped oscillation of the disk**

1. Use the disk as in Exercise 2 and let it oscillate for about 10-15 s (make sure that it is not completely damped).

2. With the DataStudio function FFT (Fast Fourier Transform) determine the frequency f of damped oscillation of the disk. Figure 7 is an example of an FFT spectrum where the placement of the peak on the x-axis yields the frequency of oscillation.

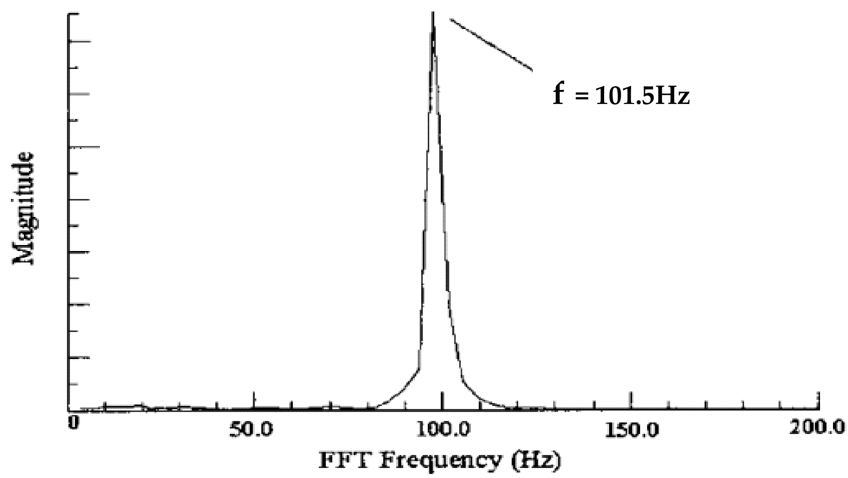

FIG.: 7

3. Calculate β, which is the amount of time that it takes for the initial value to drop by $1/e$ or 37%. [e = 2.71828… is the base of natural logarithm, or *Euler's constant*]
   β can be calculated in two different ways:

   - Method 1: Using the smart tool on your Angular position vs. Time, calculate amplitude at $t = 0$. Multiply this value by $1/e$. Find the new value of the amplitude on this Angular position vs. Time graph. Record the time that this new amplitude occurs at. This is your value of β.
   - Method 2: On the damped oscillations graph (Fig.8) determine instants of time separated by one period of oscillation and corresponding to the current maximum deviation of the disk from its equilibrium position. You have got a table of amplitude as a function of time. DataStudio has no option for exponential function fit. However, if you convert amplitude into a specific function of the amplitude, the linear fit will become applicable. Make the conversion, fit your data and find $\alpha = 1/\beta$.

4. Using the value I_{D1} and α, calculate the damping coefficient c according to Eq.7 (method of amplitudes). With I_{D1} obtain the value of ω_0 from Eq. 3* and substitute it into Eq.3** to find the other value of c (method of frequencies). Compare the two values of damping coefficient. Which one is more accurate and why?

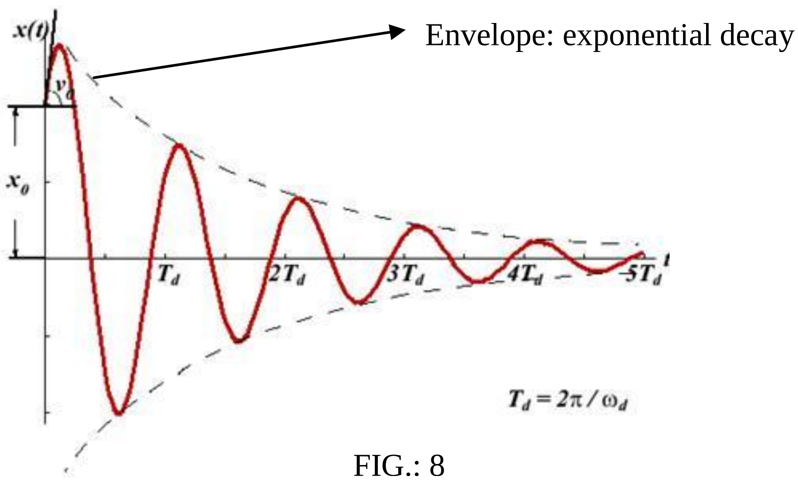

FIG.: 8

Exercise 6: **Comparing damping in wires with different diameter *d***

Repeat Exercises 1 and 5 for the other two wires. Compare the values of c, α and Q for three wires using equation $\alpha = \dfrac{\omega_0}{2Q}$. What is the trend for relationships c vs. d and α vs. d ?

The damped oscillations with $Q < ½$ are called overdamped; they stop immediately without actual oscillating. The damped oscillations with $Q = ½$ are called critically damped; they stop within one cycle. The damped oscillations with $Q > ½$ are called underdamped and can continue as shown in Fig.8. What kind of damped oscillation did you observe for three wires? Taking into account that experiments with torsional oscillation are performed with same oscillating object (a disk) and wires with different diameters, what makes the main contribution to the damping phenomenon: dissipation of mechanical energy in crystal lattice of the wire or air resistance to oscillation of the disk?

Exercise 7: **Verifying the relationship between the torsion constant and the diameter of a wire (see Eq.4 on page 2)**

a) Plot a graph of the torsion constant κ vs. (d^4/l) on DataStudio. Perform a linear fit and calculate an average value of shear modulus G according to Eq.4.

b) The value of shear modulus G for steel is 79.3 GPa. Calculate the % difference between your measured value of G and the tabulated value of G for steel. Comment on the composition of the 3 wires.

References:

1. R.A.Serway and J.W.Jewett, Jr. Physics for Scientists and Engineers 8^{th} ed. (Thomson, 2010).

2. R.D.Knight Physics for Scientists and Engineer with Modern Physics 2^{nd} ed. (Pearson, 2008).

3. P. Mohazzabi and B. M. Shefchik. “A Universal Relationship between Spring Constant and Torsion Constant.” *Journal of Physics and Chemistry of Solids* 62 (2001): 677-81. *Elsevier*. Web. 26 May 2011. <http://www.elsevier.nl/locate/jpcs>.

4. J. B. Vise. Mechanical Oscillations-Resonance and Ringing in a Tuning Fork. University of Toronto, ON (2000). <http://faraday.physics.utoronto.ca/IyearLab/tunfk.pdf>.

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