

Convolution

In this chapter and the next it will seem that this *physics* course is actually just mathematics and definitions of words that have little to do with the real world. If you bear with me, we shall emerge on the other side to do some physics.

■ Introduction

Convolution is sometimes called *Faltung* which is German for *folding*, and is also described by terms such as *running mean*, *cross-correlation function*, *smoothing*, and so on.

The convolution of two functions $f(t)$ and $g(t)$ is:

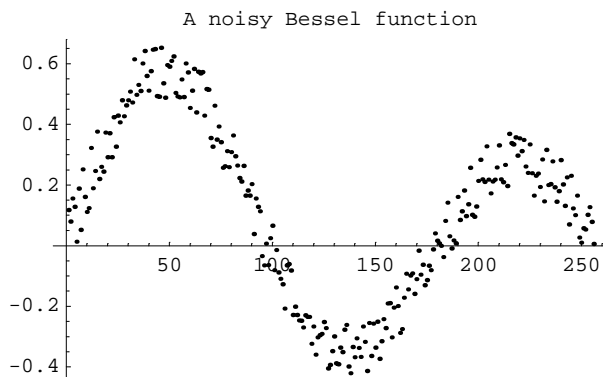
$$h(t) \equiv \int_{-\infty}^{\infty} f(u) g(t-u) du$$

Often we shall write this as:

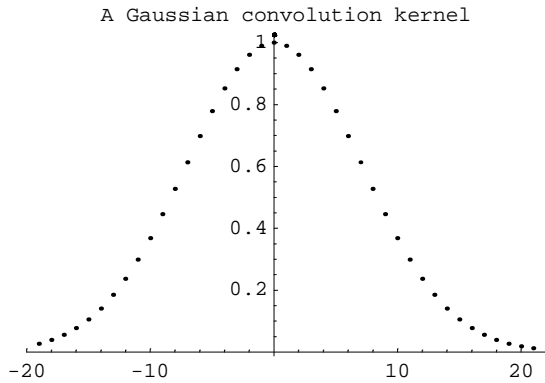
$$h(t) = f(t) * g(t)$$

The above form is strictly notational. Do not use it, for example, with *Mathematica* which will interpret the asterisk as multiplication.

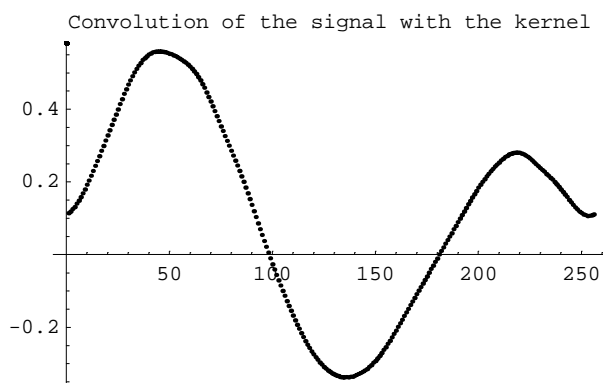
Convolution turns out to be amazingly useful for a number of tasks. Here is one of them. We have a noisy Bessel function signal:



We will convolve this signal with a Gaussian "kernel":



The result of the convolution smooths out the noise in the original signal:



You will notice that in the above example, the signal and the kernel are both discrete time series, not continuous functions. In this case, the convolution is a sum instead of an integral:

$$\mathbf{h}_i = \sum_{j=0}^m \mathbf{f}_j \mathbf{g}_{i-j}$$

Here is an example. Choose \mathbf{f} and \mathbf{g} to be:

$$\mathbf{f} = \{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\}$$

$$\mathbf{g} = \{\mathbf{g}_0, \mathbf{g}_1\}$$

Then:

$$\mathbf{h}_0 = \sum_{j=0}^m \mathbf{f}_j \mathbf{g}_{0-j}$$

The \mathbf{f}_j terms are non-zero only for j between 0 and 2, so m is equal to 2. Also, \mathbf{g}_{0-j} is non-zero only for j equal to zero. So:

$$\mathbf{h}_0 = \mathbf{f}_0 \mathbf{g}_0$$

Similarly:

$$\mathbf{h}_1 = \sum_{j=0}^m \mathbf{f}_j \mathbf{g}_{1-j} = \mathbf{f}_0 \mathbf{g}_1 + \mathbf{f}_1 \mathbf{g}_0$$

$$\mathbf{h}_2 = \sum_{j=0}^m \mathbf{f}_j \mathbf{g}_{2-j} = \mathbf{f}_1 \mathbf{g}_1 + \mathbf{f}_2 \mathbf{g}_0$$

$$\mathbf{h}_3 = \sum_{j=0}^m \mathbf{f}_j \mathbf{g}_{3-j} = \mathbf{f}_2 \mathbf{g}_1$$

Thus:

$$\mathbf{h} = \mathbf{f} * \mathbf{g} = \{\mathbf{f}_0 \mathbf{g}_0, \mathbf{f}_0 \mathbf{g}_1 + \mathbf{f}_1 \mathbf{g}_0, \mathbf{f}_1 \mathbf{g}_1 + \mathbf{f}_2 \mathbf{g}_0, \mathbf{f}_2 \mathbf{g}_1\}$$

Now we form the convolution:

$$\mathbf{h}' = \mathbf{g} * \mathbf{f}$$

Then:

$$(\mathbf{h}')_0 = \sum_{j=0}^m \mathbf{g}_j \mathbf{f}_{0-j}$$

where now, since \mathbf{g}_j is non-zero only for values of j less than or equal to 2, m is equal to 2. The result is:

$$(\mathbf{h}')_0 = \mathbf{g}_0 \mathbf{f}_0$$

This is the same as \mathbf{h}_0 calculated above. Similarly, it is simple to show that all the terms in \mathbf{h}' are the same as \mathbf{h} . In general for all convolutions:

$$\mathbf{f} * \mathbf{g} = \mathbf{g} * \mathbf{f}$$

Before we described convolving a noisy Bessel function signal with a Gaussian kernel. We now see that we could equally well say we were convolving a Gaussian signal with a noisy Bessel function kernel.

It can also be shown that in general:

$$\mathbf{f} * (\mathbf{g} * \mathbf{h}) = (\mathbf{f} * \mathbf{g}) * \mathbf{h}$$

$$\mathbf{f} * (\mathbf{g} + \mathbf{h}) = \mathbf{f} * \mathbf{g} + \mathbf{f} * \mathbf{h}$$

Here is a way to visualise how the convolution calculations are done. Imagine we have two strips of paper, and on one we write the elements of \mathbf{f} and on the other we write the elements of \mathbf{g} in reverse order.

f_0	f_1	f_2	f_3	...	f_m
g_n	...	g_3	g_2	g_1	g_0

To get the first element of the convolution, we put the left-most element of \mathbf{f} above the right-most element of \mathbf{g} :

					f_0	f_1	f_2	f_3	...	f_m
g_n	...	g_3	g_2	g_1	g_0					

Then we read out the value of \mathbf{h}_0 as the product of the overlapping terms: $f_0 g_0$.

To get the next term we overlap two of the values:

					f_0	f_1	f_2	f_3	...	f_m
g_n	...	g_3	g_2	g_1	g_0					

The result, \mathbf{h}_1 , is the sum of the products of the overlapping terms: $f_0 g_1 + f_1 g_0$.

The next term in the convolution comes by overlapping three values:

					f_0	f_1	f_2	f_3	...	f_m
g_n	...	g_3	g_2	g_1	g_0					

This gives $f_0 g_2 + f_1 g_1 + f_2 g_0$.

We continue shifting the strips by one position and reading out the result.

Finally, here is a representation of the last term in the convolution:

f_0	f_1	f_2	f_3	...	f_m					
					g_n	...	g_3	g_2	g_1	g_0

The value is, of course, $f_m g_n$.

The above technique of visualising convolution calculations makes it fairly easy to see that the number of terms in the convolution of \mathbf{f} and \mathbf{g} is the number of terms in \mathbf{f} plus the number of terms in \mathbf{g} minus 1.

$$\text{Length}[\mathbf{h}] = \text{Length}[\mathbf{f}] + \text{Length}[\mathbf{g}] - 1$$

We end this discussion with a final example of convolution. Let:

$$\mathbf{f} = \{f_0, f_1, f_2, f_3, f_4 \dots, f_m\}$$

$$\mathbf{g} = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$$

Then the convolution $\mathbf{f} * \mathbf{g}$ is:

$$\mathbf{f} * \mathbf{g} =$$

$$\left\{ f_0 / 3, (f_0 + f_1) / 3, (f_0 + f_1 + f_2) / 3, (f_1 + f_2 + f_3) / 3, \dots, \right.$$

$$\left. (f_{m-2} + f_{m-1} + f_m) / 3, (f_{m-1} + f_m) / 3, f_m / 3 \right\}$$

Note that the inner terms are just the averages of three neighboring values of \mathbf{f} .

■ System Response

In this section we get closer to using convolution in a real-life situation. First we need a couple of function definitions.

The first is the Dirac delta function $\delta(t)$. This is a function that our mathematician colleagues say can not exist, although as physicists we use it routinely. It has the following properties:

$$\delta(t) = 0, t > 0$$

$$\delta(t) = 0, t < 0$$

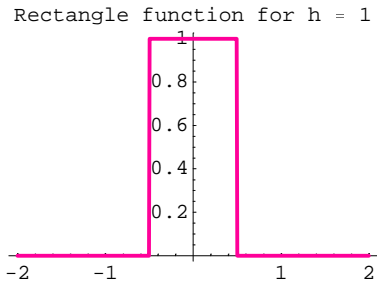
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

You may find it helpful to think of this function as something like a normalised Gaussian centered at $t = 0$ in the limit that the width of the curve σ goes to zero:

$$\lim_{\sigma \rightarrow 0} \frac{\sigma}{\sqrt{2\pi}} e^{-x^2 / 2\sigma^2}$$

The above is only a guide for your thinking: in fact the mathematicians are correct in saying this function does not really exist. Instead it is sometimes called a *improper function*. Also note that I have used e for the irrational number 2.71828...; *Mathematica* uses the symbol \mathbf{E} for this number.

Another popular way of visualising the Dirac delta function is that it is a rectangle function $\Pi(x)$ of height $1/h$ and width h in the limit that h goes to zero:



Note that the convolution of the delta function $\delta(t - t_0)$ with any function $f(t)$ gives:

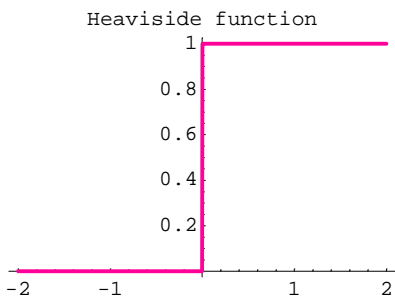
$$h(t) = f(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} f(u) \delta(u - t_0) du = f(t_0)$$

Thus, the convolution "picks" out values of f .

Consider a unit step function:

$$H(t) = 0, t < 0$$

$$H(t) = 1, t > 0$$



This is sometimes called a "Heaviside" function.

Consider the convolution:

$$a H(t - t_0) * \delta(t)$$

Here the "step" goes from zero to a at time t_0 . But what is the value of the convolution? Is it zero? Is it a ?

Probably the most reasonable choice is $a/2$, and this is the choice we will make in this course. We will make similar choices whenever we encounter a function with a discontinuity.

The convolution of the Heaviside function with any $f(t)$ is just:

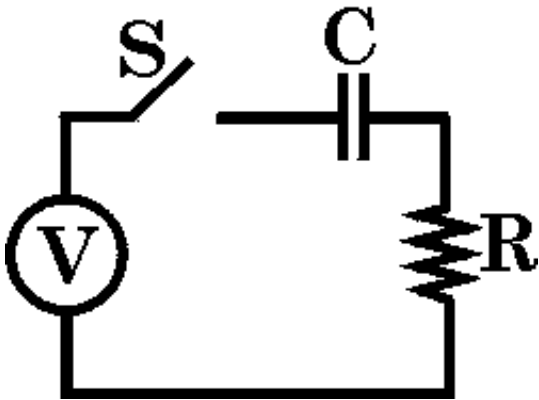
$$f(t) * H(t) = \int_{-\infty}^{\infty} f(u) H(t - u) du = \int_{-\infty}^t f(u) du$$

or:

$$f(t) = \frac{d}{dt} (f(t) * H(t))$$

It is also interesting to note the the derivative of \mathbf{H} with respect to \mathbf{t} looks like the Dirac delta function. In fact, if we ignore the fact that the delta function is *improper*, we can and will say that the derivative of \mathbf{H} is the delta function.

Now we consider a simple electrical circuit:



At time t equal to zero the capacitor is uncharged and the switch \mathbf{S} is then closed. Thus the input voltage is:

$$V_{\text{inp}}(t) = V H(t)$$

It is simple to show that the current \mathbf{I} is:

$$I = \frac{V}{R} e^{-t/RC} H(t)$$

We define the *impulse response* of the circuit as the values of the current when the input voltage is:

$$V_{\text{inp}}(t) = V \delta(t)$$

The impulse response is just the derivative of the current \mathbf{I} to the original input voltage:

$$I_{\text{impulse}} = \frac{d}{dt} \left(\frac{V}{R} e^{-t/RC} H(t) \right) = \frac{V}{R} \left(\delta(t) - \frac{1}{RC} e^{-t/RC} H(t) \right)$$

In fact, it can be shown that if the impulse response of *any* system is $\mathbf{R}_{\text{impulse}}$, then its response to any input \mathbf{Inp} is the convolution:

$$\mathbf{Inp} * \mathbf{R}_{\text{impulse}}$$

Finally, a little nomenclature regarding systems.

We say a system is *linear* if its response to any input is proportional to the magnitude of the input to the system. Prof. Drummond characterises a linear system this way: "If I kick it it screams; if I kick it twice as hard it screams twice as loud."

Finally, we are assuming that the input to the system begins at time $t = 0$. If the output from the system for negative times is also zero, we say the system is *causal*. Later you may be surprised to learn that we sometimes consider non-causal systems.

Usually, however, we shall be considering linear causal systems in this course.

■ Author

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