Lecture 17

Mechanical impedance

Examples

References:
Hirose Ch.6 (6.4)
or:
Lecture notes

Incomplete reflection

Reflection at a free boundary and reflection at a fixed boundary are two ideal cases. In either case, reflection is complete in a sense that all energy associated with an incident wave is reflected. In practice complete reflection rarely happens.

Traveling mechanical waves can be expressed by the general function:

$$\xi(x, t) = f(x \pm c_w t)$$

Force is proportional to the space derivative of the wave function.

In a string: $F = -T \sin\theta = -T \frac{\partial \xi}{\partial x}$ (transverse wave)

For a transmission line: $F = -k_s \frac{\partial \xi}{\partial x} \Delta x = -K \frac{\partial \xi}{\partial x}$ (longitudinal wave)

Velocity wave is given by: $v = \frac{\partial \xi}{\partial t} = \pm c_w \dot{f}(x \pm c_w t)$

If we combine force and velocity:

$$\frac{F}{v} = -\frac{(T \text{ or } K)}{\pm c_w} = \mp\sqrt{(T \text{ or } K)\rho_\ell}$$

$$c_w = \sqrt{\frac{(T \text{ or } K)}{\rho_\ell}}$$

We define Z (mechanical impedance) as:

$$Z = \sqrt{(T \text{ or } K)\rho_\ell}$$

$$F = \pm Zv \begin{cases} \text{"+" for forward - going waves} \\ \text{"-" for backward - going waves} \end{cases}$$

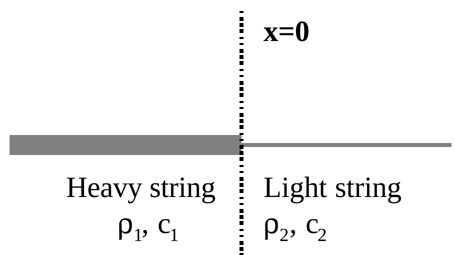
T is tension for transverse waves; K is the elastic modulus for longitudinal waves

In mechanical waves the force F(x, t) and the velocity v(x, t) are always proportional to each other

Connection between two media

To illustrate incomplete reflection, we consider two strings with different mass densities ρ_1 and ρ_2 both subject to a common tension T. The velocities of transverse waves on each string are c_1 and c_2.

At the connection between two media:



The energy flow associated with a sinusoidal wavetrain with amplitude ξ_0 and frequency $\omega = 2\pi\nu$ along a string with mass density ρ_l is given by:

$$\left\langle \frac{dE}{dt} \right\rangle = c_w \left\langle \frac{dE}{dx} \right\rangle = \frac{1}{2} c_w \rho_\ell \omega^2 \xi_0^2$$

We asume that the incident wave has an amplitude ξ_1, the reflected wave ξ_r and the transmitted ξ_2 , respectively. The energy flow and the direction (little arrow) associated with:

- the incident wave is: $\frac{1}{2} c_1 \rho_1 \omega^2 \xi_1^2$ ®

- the reflected wave: $\frac{1}{2}c_1\rho_1\omega^2\xi_r^2$ ¬

- the transmitted wave: $\frac{1}{2}c_2\rho_2\omega^2\xi_2^2$ ®

From the principle of energy conservation, we must have:

$$\frac{1}{2}c_1\rho_1\omega^2\xi_1^2 - \frac{1}{2}c_1\rho_1\omega^2\xi_r^2 = \frac{1}{2}c_2\rho_2\omega^2\xi_2^2$$

or:

$$\boxed{c_1\rho_1(\xi_1^2 - \xi_r^2) = \rho_2 c_2\xi_2^2}$$

The displacement at the boundary has to be continuous which means:

$$\boxed{(\xi_1 + \xi_r) = \xi_2}$$

The two simultaneous equations above allow ξ_r and ξ_2 to be calculated:
Solving these, we find:

$$\xi_r = \frac{\sqrt{\rho_1 T} - \sqrt{\rho_2 T}}{\sqrt{\rho_1 T} + \sqrt{\rho_2 T}}\xi_1 \quad \text{and :}$$

$$\xi_2 = \frac{2\sqrt{\rho_1 T}}{\sqrt{\rho_1 T} + \sqrt{\rho_2 T}}\xi_1$$

which can be re-written by using the mechanical impedance of each medium:

$$\xi_r = \frac{Z_1 - Z_2}{Z_1 + Z_2}\xi_1 \quad \text{and :}$$

$$\xi_2 = \frac{2Z_1}{Z_1 + Z_2}\xi_1$$

The condition for the absence of wave reflection at a boundary between two media is that the impedance of the media be the same

We can calculate now, for instance, the fraction of reflected wave energy:

$$\left(\frac{\xi_r}{\xi_1}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

where **$[(Z_1 - Z_2)/(Z_1 + Z_2)]^2$ is the coefficient of reflection.**