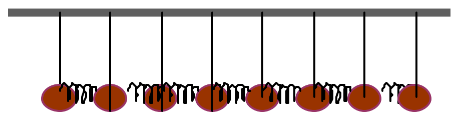
Final Review - part II

Going continuous: from oscillators to waves
What it takes to create a wave
What keeps a wave moving
Longitudinal vs. transverse waves
Force in a transverse wave
Boundary conditions; reflection
Standing waves

Going continuous: from oscillators to waves



We made N very large while making m and the spring smaller and smaller (it will look like a spring with distributed mass).

THEN:

We changed the variable notation from x_n to ξ_n, and we assigned each individual mass to a position along the mass-spring line:

$$\xi_n \rightarrow \xi(x)$$
$$\xi_{n-1} \rightarrow \xi(x - \Delta x)$$
$$\xi_{n+1} \rightarrow \xi(x + \Delta x)$$

AND

We defined:

- $\frac{m}{\Delta x}$ = linear (mass) density $\rho_\ell \left(\frac{kg}{m}\right)$

- **the elastic modulus K:**

$K = k_s \Delta x$ where k_s is the elastic constant.

K refers to a finite object.

We wrote the **wave equation**:

$$\boxed{\rho_\ell \frac{\partial^2}{\partial t^2}\xi(x,t) = K\frac{\partial^2}{\partial x^2}\xi(x,t)}$$ for a massive uniform spring

How to get the solution to wave equation:

We extended the solution of n-coupled oscillators equations:

Equation of motion: $$m\frac{d^2}{dt^2}\xi_n = -k(\xi_n - \xi_{n-1}) - k(\xi_n - \xi_{n+1})$$

OR:

$$m\frac{d^2\psi}{dt^2} = M\psi$$ where:

$$\psi = \begin{pmatrix} \vdots \\ \xi_{n-1} \\ \xi_n \\ \xi_{n+1} \\ \vdots \end{pmatrix} \text{ and } M = \begin{pmatrix} \ddots & & & & \\ \ddots & -2k & k & & \\ & k & -2k & k & \\ & & k & -2k & k \; \ddots \\ & & & & \ddots \end{pmatrix}$$

The oscillation should look like: $$\psi = \begin{pmatrix} \vdots \\ \xi_{n-1} \\ \xi_n \\ \xi_{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ a_{n-1} \\ a_n \\ a_{n+1} \\ \vdots \end{pmatrix} e^{i\omega t}$$

By extension, the solution for the continuous wave equation should look like:

$$\boxed{Z(x,t) = a(x)e^{\pm i\omega t}}$$

OR

$$\boxed{\xi(x,t) = \xi_0 \cos(kx \pm \omega t)}$$

the usual form

Fundamental constants:

$$k = \omega\sqrt{\frac{\rho_\ell}{K}} \quad \text{the wave number}$$

$$\lambda = \frac{2\pi}{k} \quad \text{the wavelength}$$

$$c_w = \mp\sqrt{\frac{K}{\rho_\ell}} \quad \text{the wave velocity}$$

What it takes to create waves:

Power = Energy transfer rate
Force = Momentum transfer rate

Waves carry momentum and energy. Motor must give force and power to create waves.

Energy is distributed non-uniformly over space
It travels at the wave velocity

Momentum is also distributed non-uniformly over space
It also travels at the wave velocity

Problems with longitudinal waves:

1) Mass density varies → it changes momentum
2) Density variation changes the wave number → this changes the force

Other "waves":

Density variation wave:

$$\Delta\rho_{\ell} = \rho_{\ell}\xi_0 k \sin(kx - \omega t)$$

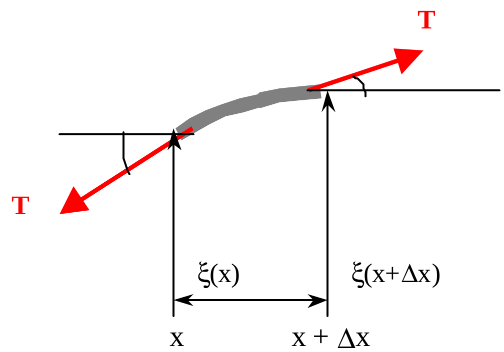
Velocity wave:

$$v(x,t) = \frac{\partial \xi(x,t)}{\partial t} = \xi_0 \omega \sin(kx - \omega t)$$

Force wave:

$$F(x,t) = -K\left(\frac{\partial \xi(x,t)}{\partial x}\right) = -K\xi_0 k \sin(kx - \omega t)$$

Force of a transverse wave in a string



The tension T stretches the string

T at x has a slope whose direction is given by:

$$\tan\theta = \left.\frac{\partial \xi}{\partial x}\right|_{\text{at } x}$$

$$F \cong T(\theta_2 - \theta_1) = T\left(\left.\frac{\partial \xi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial \xi}{\partial x}\right|_{x} \right) \cong \Delta x T \frac{\partial^2 \xi}{\partial x^2}$$

This is the force on a segment Δx

Energy and momentum carried by transverse waves are identical to those of longitudinal waves.

Reflection of waves

At a fixed boundary the total displacement ξ stays zero and the reflected wave changes its polarity.

At a free boundary, the restoring force is zero or $\frac{\partial \xi}{\partial x} = 0$ and the reflected wave has the same polarity as the incident wave. The amplitude at the free boundary is twice as large as that of the incident wave

The resultant standing wave (sum of the original and reflected waves) is:

$$\xi_0 \cos(kx - \omega t) \pm \xi_0 \cos(kx + \omega t) \rightarrow \begin{cases} 2\xi_0 \cos kx \sin \omega t \ (\text{free end}) \\ 2\xi_0 \sin kx \cos \omega t \ (\text{fixed end}) \end{cases}$$

Wave length of the standing wave:

$$\lambda = \frac{2\pi}{k}$$

Incomplete reflection

In mechanical waves the force F(x, t) and the velocity v(x, t) are always proportional to each other

We define Z (mechanical impedance) as:

$$Z = \sqrt{(T \text{ or } K)\rho_\ell}$$

➔ $$F = \pm Zv \begin{cases} \text{"+" for forward - going waves} \\ \text{"-" for backward - going waves} \end{cases}$$

T (tension) for transverse waves; K (elastic modulus) for longitudinal waves

$$\xi_{\text{reflected}} = \frac{Z_1 - Z_2}{Z_1 + Z_2}\xi_{\text{incident}}$$

$$\xi_{\text{transmitted}} = \frac{2Z_1}{Z_1 + Z_2}\xi_{\text{incident}}$$

$$c_1\rho_1(\xi_i^2 - \xi_r^2) = \rho_2 c_2 \xi_t^2$$ **conservation of energy**

$$(\xi_i + \xi_r) = \xi_t$$ **continuity at the boundary**