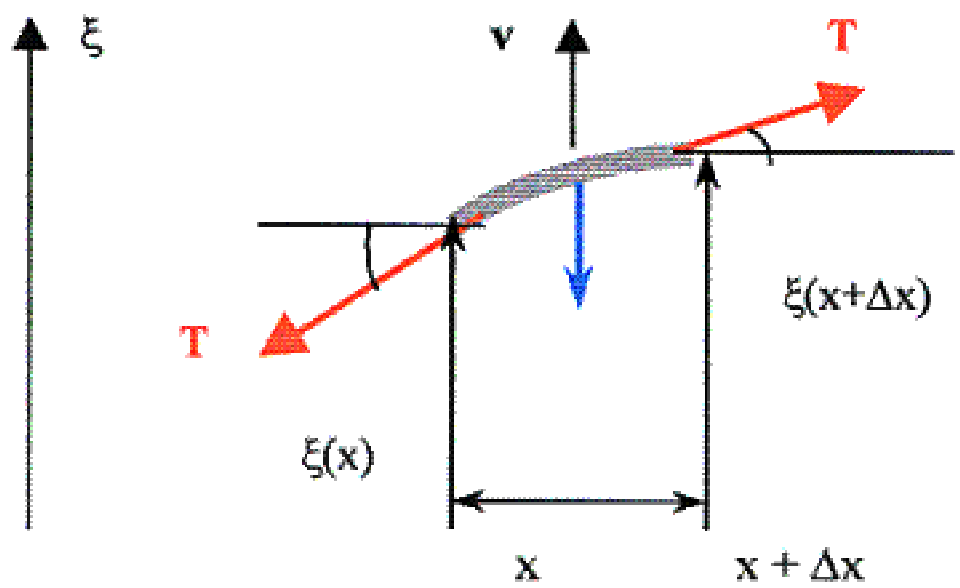
Example for Lecture 15



Consider a chunk of string of length Δx and mass per unit length ρ_l. The string is under tension T. The displacement in the vertical direction is given by the variable $\xi(x, t)$.
We now include the air resistance. If the string is moving upwards with velocity **v**, the air resistance creates a force opposing the direction of velocity: $\mathbf{F}_{drag} = -b\, m_c\, \mathbf{v}$. where b is a positive constant and m_c is the mass of the chunk.
We ignore the small effects of gravity.

a) Write the equation of motion in the presence of air resistance;
b) Calculate the kinetic energy density of the string with no air resistance
c) The same piece of string has a potential energy due to its elongation. Calculate the potential energy density. Ignore the air resistance.

a) The drag force is vertical, so the eq. of motion should involve the "y" component of the velocity : $v_y \approx v = \frac{\partial \xi(x,t)}{\partial t}$

$$m_c a_y = F_{net} = -\underbrace{T\frac{\partial^2 \xi}{\partial x^2}}_{\text{see Lecture 15 notes}} - b\, m_c \frac{\partial \xi}{\partial t}$$

$$a_y = \frac{\partial^2 \xi}{\partial t^2}$$

$$m_c \frac{\partial^2 \xi}{\partial t^2} + b\, m_c \frac{\partial \xi}{\partial t} = -T\frac{\partial^2 \xi}{\partial x^2}$$

b) Kinetic energy density is $E_K = \frac{m_c v^2}{2}$

where $m = \rho_l \Delta x$; $v(x,y) = \frac{\partial \xi(x,t)}{\partial t}$

$$E_K = \rho_l \Delta x \left(\frac{\partial \xi}{\partial t}\right)^2 = \rho_l \Delta x\, \omega^2 \xi_0^2 \sin^2(kx - \omega t)$$

if we used $\xi = \xi_0 \cos(kx - \omega t)$

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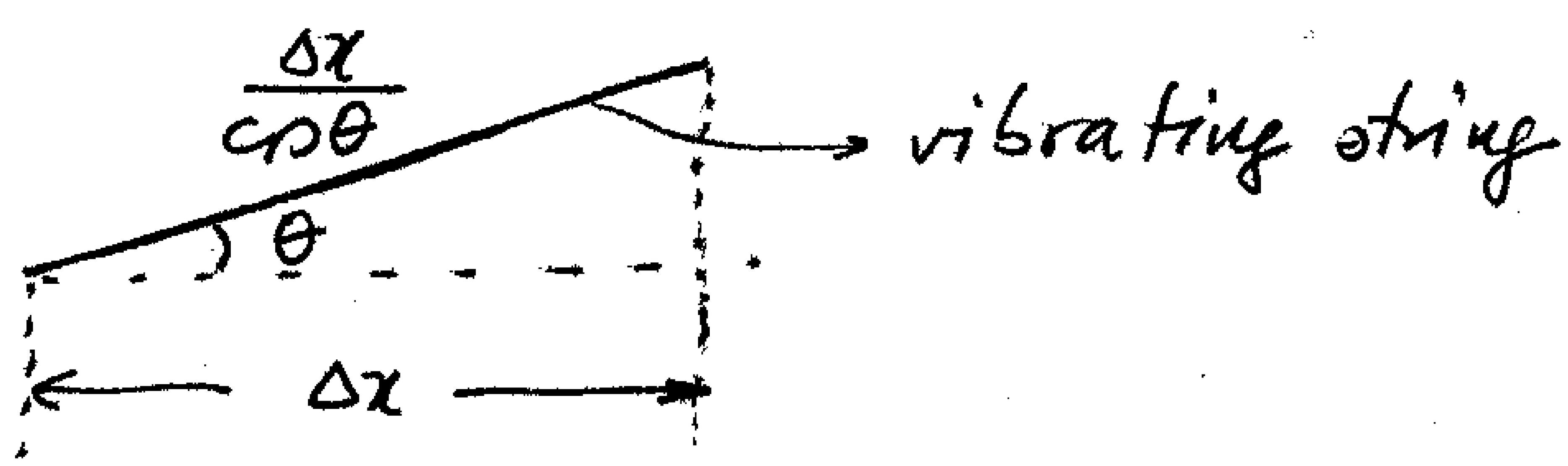
$$\frac{dE_K}{dx} = \rho_\ell \omega^2 \xi_0^2 \sin^2(kx - \omega t)$$

b) Potential energy due to elongation



Actual length due to vibration is $\frac{\Delta x}{\cos\theta}$

Stretch is $\left(\frac{\Delta x}{\cos\theta} - \Delta x\right)$

Potential energy $E_p = T\left(\frac{\Delta x}{\cos\theta} - \Delta x\right) =$

$$= T\Delta x\left(\frac{1}{1 - \frac{\theta^2}{2}} - 1\right) \approx T\Delta x\left(1 + \frac{\theta^2}{2} - 1\right) \approx T\Delta x\frac{\theta^2}{2}$$

$$\frac{dE_p}{dx} = \frac{T}{2}\theta^2 \quad , \text{ but } \theta \approx \tan\theta \approx \frac{\partial \xi}{\partial x}$$

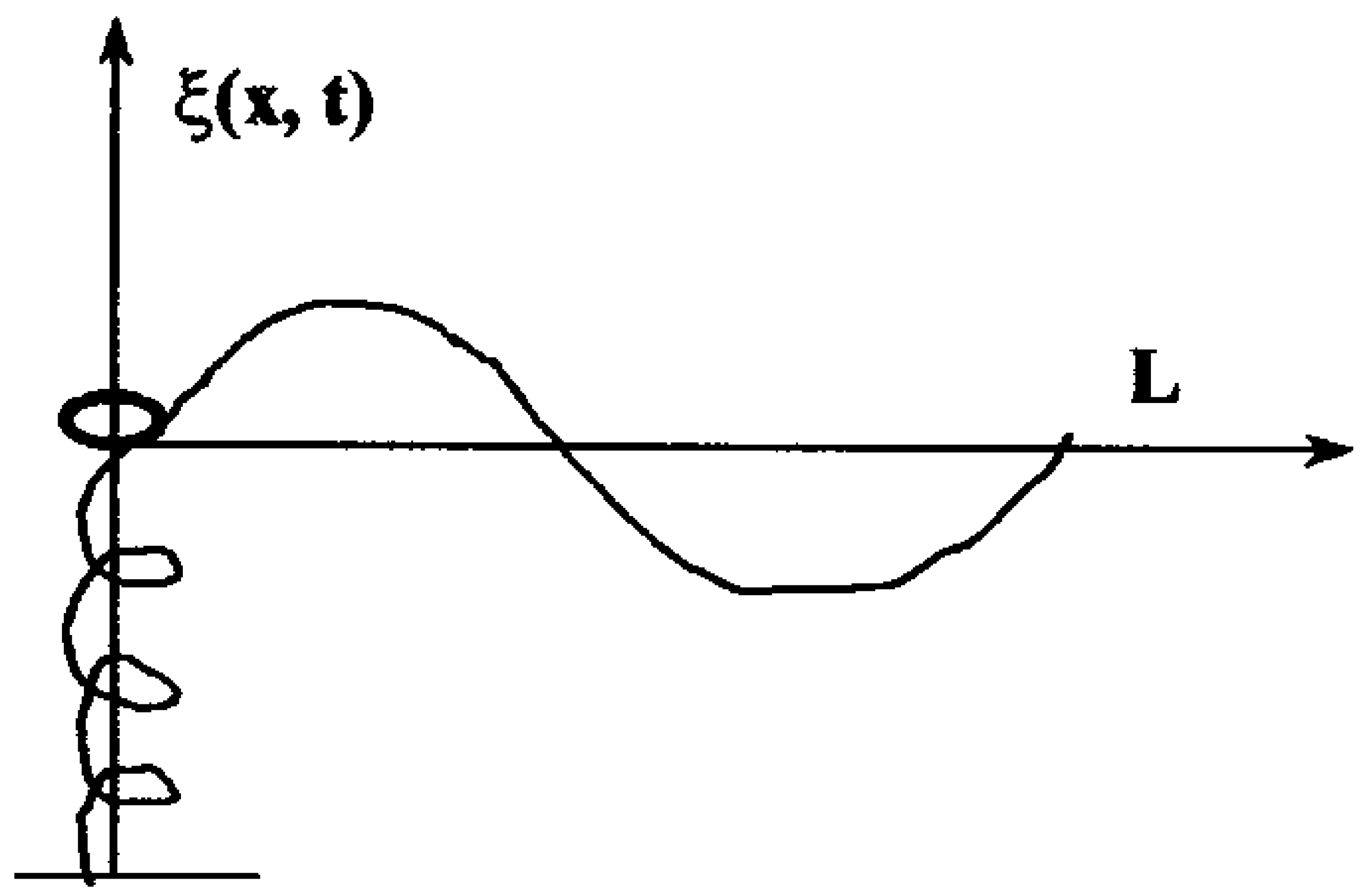
$$\frac{dE_p}{dx} = \frac{T}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{T}{2}\xi_0^2 k^2 \sin^2(kx - \omega t)$$

with $k^2 = \omega^2 \frac{\rho_\ell}{T}$ we get $\frac{dE_p}{dx} = \frac{\rho_\ell}{2}\omega^2 \xi_0^2 \sin^2(kx - \omega t)$

$$\frac{dE_p}{dx} = \frac{dE_K}{dx}$$

Example for Lecture 16

A string of length L is under tension T and has linear mass density μ (this is mass per unit of length). The string is fixed at one end (x = L) while the other end (x = 0) is attached to a massless ring that can slide freely (without friction) up and down a rod. Attached to the ring is an ideal spring that exerts a vertical force on the ring; when the ring is at y = 0 the spring is relaxed (see the figure below).



In this problem K_{spr} represents the spring constant to avoid confusion with the wave number $k = 2\pi/\lambda$. If the spring is extremely stiff (K_{spr} is very large), x = 0 acts like a fixed end. If the spring is extremely flexible (K_{spr} is very small), x = 0 acts like a free end.

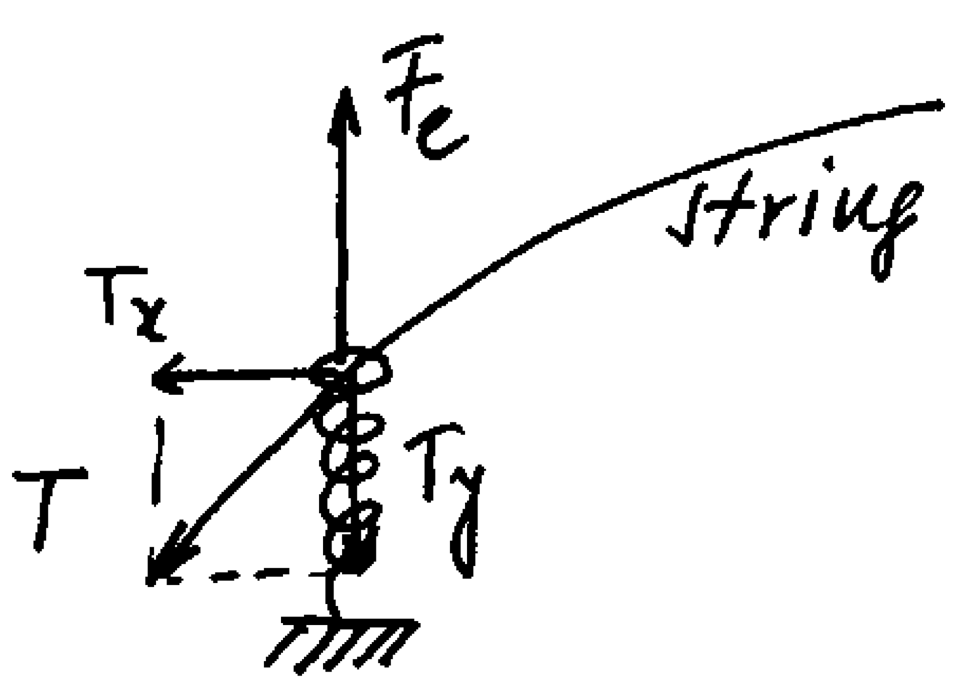
a) Use Newton's Second Law to find a relationship between the slope of the string at x = 0 and the displacement $\xi(0, t)$ of that point (which is the same as the displacement of the ring).

b) Now let the spring constant have an arbitrary value. Would any of the standing wave solutions for a free end or a fixed end work for the general boundary condition you derived in a)? Demonstrate that a standing wave of the form:

$$\xi(x,t) = A\cos(kx + \phi')\cos(\omega t) \qquad (1)$$

would work in this case. Find the condition(s) that the initial phase ϕ' and the wavelength λ should satisfy if the standing wave from equation (1) is to be formed on the string.

a)



T = tension in the string

F_e = elastic force of the spring

$$T_y = -T\frac{\partial \xi}{\partial x}\Big|_{x=0}$$

$$F_e = -K_{spr}\,\xi(0,t)$$

Net force on Oy: $F_{y,net} = -K_{spr}\,\xi(0,t) - T\frac{\partial \xi}{\partial x}\Big|_{0,t}$

Newton's second law: $m_{ring}\,a_y = F_{y,net}$

but $m_{ring} = 0$

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Ex_ lect 16

$$\underbrace{K_{spr}\,\xi(0,t)}_{\text{displacement at } x=0} = -T \underbrace{\left.\frac{\partial \xi}{\partial x}\right|_{x=0}}_{\text{slope at } x=0} \qquad (1) \text{ Boundary condition}$$

b) If $K_{spr} \to \infty$, $x=0$ is a fixed end

If $K_{spr} \to 0$, $x=0$ is a free end

K_{spr} arbitrary $\Rightarrow$ use boundary cond. (1)

for $\xi(x,t) = A\cos(kx+\varphi')\cos\omega t$

$$-TkA\sin(kx+\varphi')\cos\omega t = K_{spr}A\cos(kx+\varphi')\cos\omega t$$

$$\tan(kx+\varphi') = -\frac{K_{spr}}{kT}$$

$$kx+\varphi' = -\tan^{-1}\left(\frac{K_{spr}}{kT}\right)$$

at $x=0$, $\qquad \varphi' = -\tan^{-1}\left(\frac{K_{spr}}{kT}\right) \qquad (2)$

Use boundary condition at $x=L$ (clamped end):

$$\xi(L,t) = A\cos(kL+\varphi')\cos\omega t = 0$$

$$\Rightarrow \quad (kL+\varphi') = \frac{2n-1}{2}\pi \qquad (3)$$

Combine (2) and (3) :

$$kL - \tan^{-1}\left(\frac{K_{spr}}{kT}\right) = n\pi - \frac{\pi}{2}$$

where $k = \frac{2\pi}{\lambda}$ (the wave number)

$$\frac{2\pi}{\lambda}L - \tan^{-1}\left(\frac{K_{spr}\lambda}{2\pi T}\right) = n\pi - \frac{\pi}{2} \qquad (4)$$

continued →

Ex_ Lect 16

Analyze (4) in terms of K_{spr}:

1) $K_{spr} \longrightarrow \infty \implies \tan^{-1}\left(\frac{K_{spr}\lambda}{2\pi T}\right) \longrightarrow \frac{\pi}{2}$

$$\frac{2\pi}{\lambda}L - \frac{\pi}{2} = n\pi - \frac{\pi}{2} \implies \boxed{\lambda = \frac{2L}{n}}$$

This is the condition for λ of a standing wave in a clamped string

2) $K_{spr} \longrightarrow 0 \implies \tan^{-1}\left(\frac{K_{spr}\lambda}{2\pi T}\right) \longrightarrow 0$

$$\frac{2\pi}{\lambda}L = n\pi - \frac{\pi}{2} \implies \boxed{\lambda = \frac{4L}{2n-1}}$$

Condition for λ of a standing wave in a string with one end free

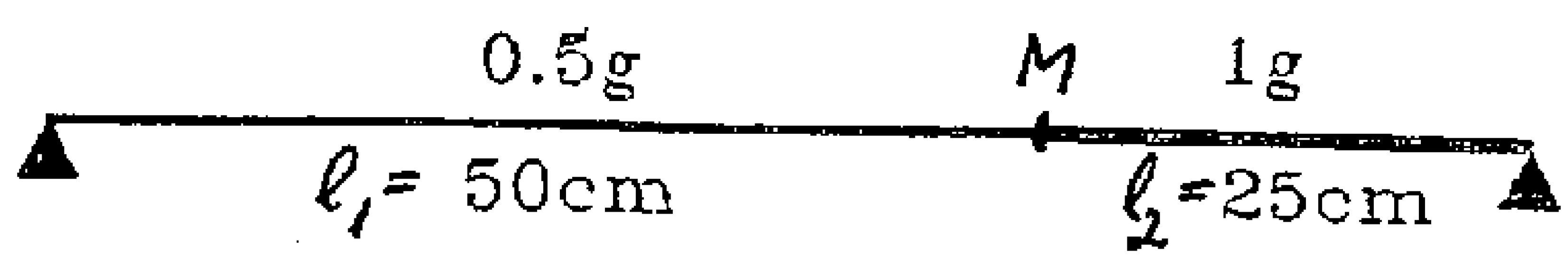
Example – Lecture 17

A string having a mass 0.5 g and length 50 cm is connected to another string of mass 1 g and length 25 cm. The strings are subject to a tension of 30 N.

(a) What is the lowest standing wave frequency for which the joint becomes a node (zero displacement)? Sketch qualitatively the standing wave pattern.

(b) What is the ratio between the amplitudes of the standing waves in each string?

(c) Find the total amount of energy associated with the standing waves when the amplitude in the lighter string is 1 mm.



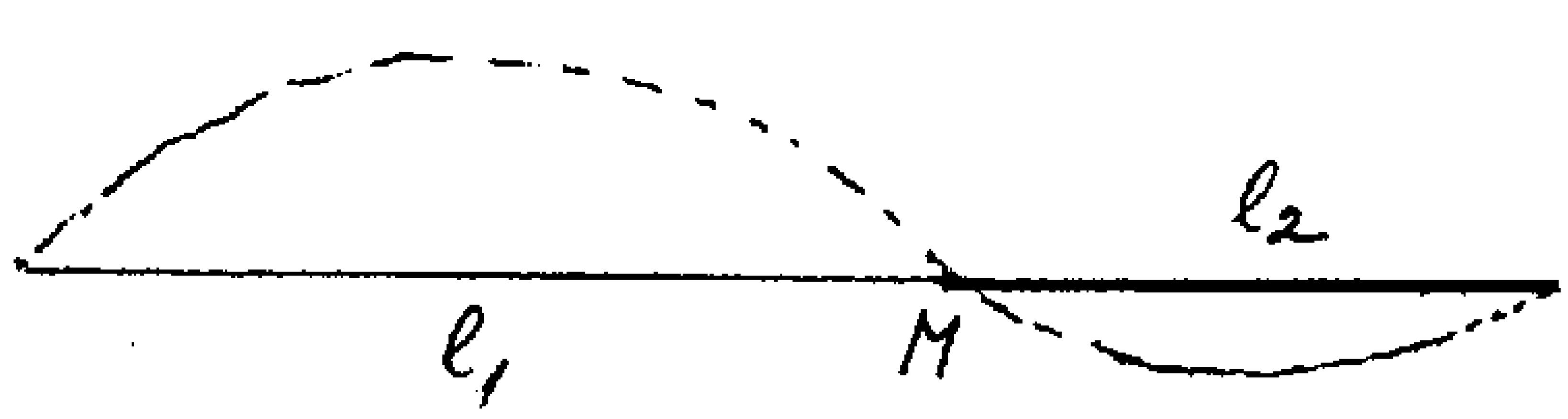
a) The wave velocities are: $c_1 = \sqrt{\frac{T}{\rho_1}}$; $c_2 = \sqrt{\frac{T}{\rho_2}}$

$$\rho_1 = \frac{m_1}{\ell_1} = 10^{-3}\ \text{Kg/m} \quad ; \quad \rho_2 = \frac{m_2}{\ell_2} = 4\times 10^{-3}\ \text{Kg/m}$$

$$\Longrightarrow \quad c_1 = 173\ \text{m/s} \quad ; \quad c_2 = 86.5\ \text{m/s}$$

For the fundamental standing wave: $\lambda_1 = \frac{2\ell_1}{1} = 1\,m$ and $\lambda_2 = \frac{2\ell_2}{1} = 0.5\,m$

Frequency is the same: $\nu = \frac{c_1}{\lambda_1} = \frac{c_2}{\lambda_2} = 173\ Hz$



The displacement $\xi(x,t)$ and its derivative $\frac{\partial \xi}{\partial x}$ must be continuous at M.

b) $\xi_1(x,t) = \xi_{01} \sin k_1 x \cos\omega t$; $k_1 = \frac{2\pi}{\lambda_1} = 2\pi\ (m^{-1})$

$\xi_2(x,t) = \xi_{02} \sin k_2 (x-\ell_1) \cos\omega t$; $k_2 = \frac{2\pi}{\lambda_2} = 4\pi\ (m^{-1})$

Continuity of derivative: $\left.\frac{\partial \xi_1}{\partial x}\right|_{x=\ell_1} = \left.\frac{\partial \xi_2}{\partial x}\right|_{x=\ell_1}$

continued →

Ex – Lect 17

$$\left.\frac{\partial \xi_1}{\partial x}\right|_{x=0.5} = k_1 \xi_{01} \cos k_1 x \cos\omega t = 2\pi \xi_{01} \cos(2\pi \times 0.5)\cos\omega t$$

$$= 2\pi \xi_{01} \cos\pi \cos\omega t = -2\pi \xi_{01} \cos\omega t$$

$$\left.\frac{\partial \xi_2}{\partial x}\right|_{x=0.5} = k_2 \xi_{02} \cos k_2 x \cos\omega t = 4\pi \xi_{02} \cos(4\pi(0.5-0.5))\cos\omega t$$

$$= 4\pi \xi_{02} \cos\omega t$$

continuity in the derivative: $-2\pi \xi_{01} = 4\pi \xi_{02}$

$$\xi_{02} = -\frac{\xi_{01}}{2}$$

c) Energy in the standing wave is: $E = E_{k1} + E_{k2}$

$$E = \frac{m_1 \langle v_1(x,t)\rangle^2}{2} + \frac{m_2 \langle v_2(x,t)\rangle^2}{2} = \frac{1}{4} m_1 (\omega \xi_{01})^2 +$$

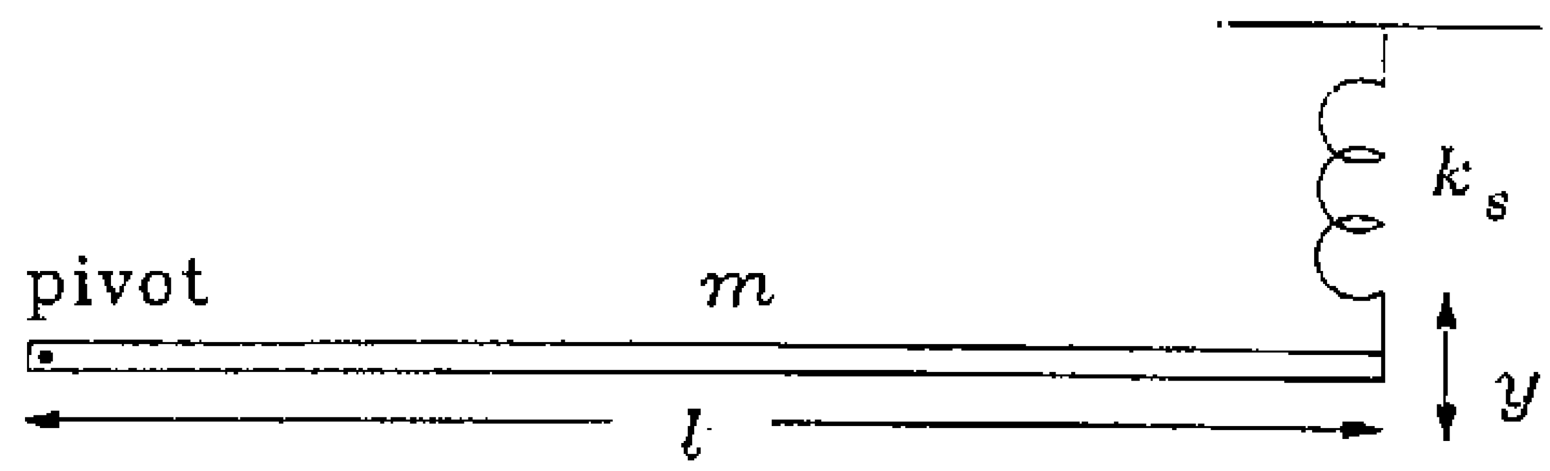
$$+ \frac{1}{4} m_2 (\omega \xi_{02})^2 = \frac{\omega^2}{4}\left(m_1 \xi_{01}^2 + \frac{m_1}{2}\xi_{01}^2\right) = \frac{3}{2}\frac{\omega^2}{4} m_1 \xi_{01}^2$$

$m_2 = 2m_1$; $\xi_{02} = -\frac{\xi_{01}}{2}$

$$E = 2.2 \times 10^{-4}\ \mathrm{J}$$

Example – Lecture 18

A rod of length l and mass m is freely pivoted at one end. Another end is attached to a mass-less spring (spring constant k_s) as shown.



(a) If the spring elongates vertically a distance y, what is the total potential energy in the system, one due to the spring elasticity and another due to gravity?

(b) What is the kinetic energy of the rod if the displacement y is varying with time?

(c) Then, using the energy conservation principle, show that the oscillation frequency is given by

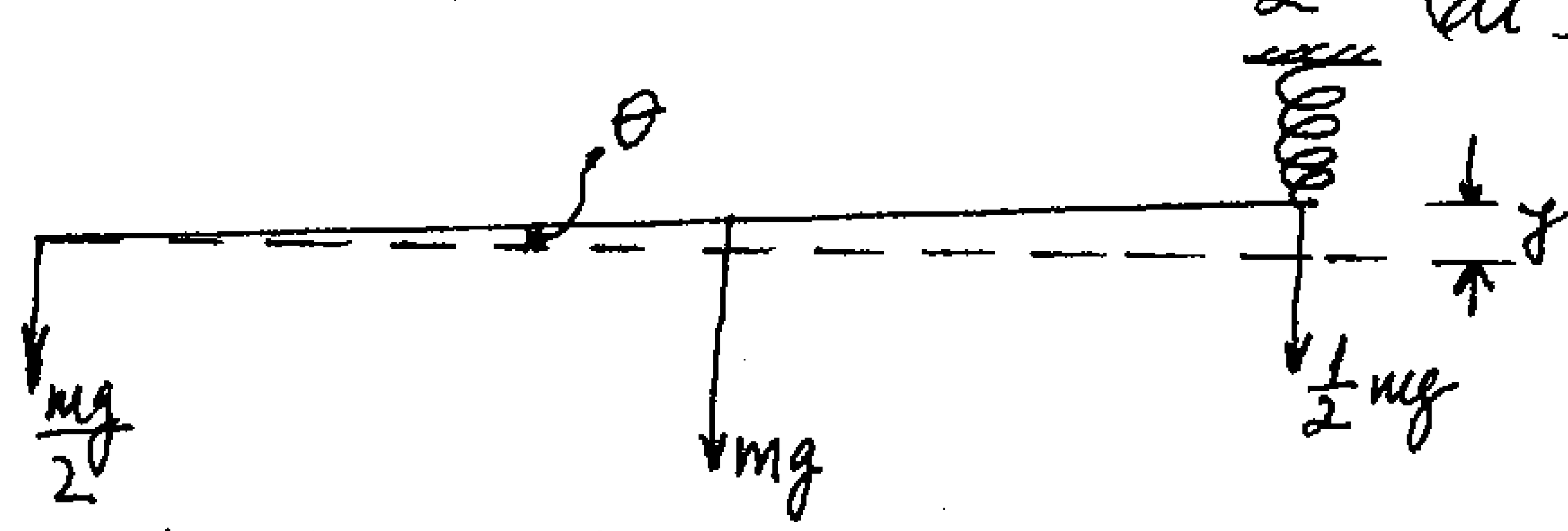
$$\omega = \sqrt{\frac{3k_s}{m}},$$

independent of both l and g.

a) Potential energy in the spring is: $\frac{k_s y^2}{2}$

Potential gravitational energy of the rod is: $mgh = \frac{mgy}{2}$

Kinetic energy of the rod is $\frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$



Moment of inertia of the rod about the end of the rod is $I = \frac{ml^2}{3}$.

Total energy must be a constant:

$$\frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + mg\frac{y}{2} + \frac{1}{2} k_s y^2 = \text{constant}$$

$$\frac{m}{6}\left(\frac{dy}{dt}\right)^2 + \frac{mgy}{2} + \frac{k_s y^2}{2} = \text{constant}$$

Differentiate with respect to time:

cont. →

Ex _ Lect 18

$$k_s y \frac{dy}{dt} + \frac{1}{2} mg \frac{dy}{dt} + \frac{1}{3} m \frac{dy}{dt} \frac{d^2 y}{dt^2} = 0$$

$$\frac{d^2 y}{dt^2} + \frac{3k_s}{m} y + \frac{3}{2} g = 0$$

The third constant term can be incorporated into a new variable : $y' = y + \frac{mg}{2k_s}$

and the equation is reduced to :

$$\frac{d^2 y'}{dt^2} + \frac{3k_s}{m} y' = 0 \quad \Longrightarrow \quad \omega = \sqrt{\frac{3k_s}{m}}$$