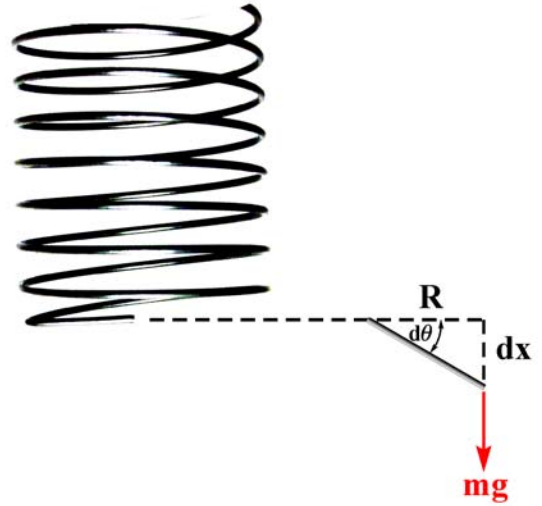


Deriving the Shear Modulus S From the Spring Constant k

When a mass is hung from the end of a spring it produces a torque or couple which has the effect of producing a uniform twist along its entire length. The stretching of the spring is due to this twisting of the spring wire along its entire length. There is a bit of stretching of the spring due to vertical shear forces but this is small and may be neglected compared to the stretching due to twisting.



Consider an element of the wire at any point on the spring. All elements are equivalent and in the diagram we consider an element at the bottom left of the spring (this element is perpendicular to the page).

When an element of spring is twisted through an angle $d\theta$, the spring which has a radius R is stretched by an amount dx where $dx = R d\theta$ as shown in the diagram. Note that the angle $d\theta$ in the diagram is exaggerated for clarity.

Integrating gives

$$\int_0^{x_{\text{total}}} dx = \int_0^{\theta_{\text{total}}} R d\theta$$

or

$$x_{\text{total}} = R \theta_{\text{total}}$$

Since this expression is true for any stretch x , we may drop the subscripts and write

$$x = R \theta$$

The energy stored in the spring can be written either in linear terms using the spring constant k

or in angular terms using the torsion constant κ given by the [formula](#) $\kappa = \frac{S \pi r^4}{2l}$ where S is the

shear modulus of the material that makes up the spring, r is the radius of the wire and l is the length of the wire that coils to form the spring. Note that κ is the torsion constant of the wire that coils to form the spring (and not the torsion constant of the spring as a whole.) Using the two expressions for energy stored in the spring gives

$$\frac{1}{2} k x^2 = \frac{1}{2} \kappa \theta^2$$

$$k x^2 = \left(\frac{S \pi r^4}{2l} \right) \left(\frac{x}{R} \right)^2$$

$$k = \frac{S \pi r^4}{2l R^2}$$

$$k = \frac{S \pi r^4}{2(2\pi R N) R^2}$$

where N is the number of turns in the coil. This gives the shear modulus in terms of the spring

constant as

$$S = k \frac{4NR^3}{r^4}$$