Solutions: **Final exam Phys. 180** **2006**

1)

i) a) All planets move around the sun in an elliptical orbit with the sun at one focus

b) The radius vector from the sun to the planet sweeps out equal areas in equal times.

c) For all planets the square of the period is proportional to the cube of the semimajor axis

ii) There are several possibilities. In a frame of reference accelerating forward there is a fictitious force equal to the negative of that acceleration times the mass of the object. In a rotating frame there is the centrifugal force, equal and opposite to centripetal force in an inertial frame, and, if the object has linear velocity in that frame, there is a coriolis force which induces a sideways acceleration proportional to the radial speed time the angular speed.

iii) A force is conservative if the work it does in moving an object from one position to another only depends on the positions of the endpoints and not the path taken.

iv) In principle, either situation could depend on speed, but in the solid/solid case the mass of one of the objects is usually considered to be infinite. Contact friction is based on rigid objects interacting with each other through effectively *static* surface forces and depends on the effective area of contact. Fluid friction is associated with <u>momentum transfer</u> from elements of the fluid to the object.

v) The Young's modulus is the ratio of stress to strain of an object for elongation or compression along an axis. Stress is the force per unit area while strain is the fractional change in length.

2) If the force of the jth particle on the ith particle is $\vec{F}_{ji}$, we have that the total force on the system is $\vec{F}_{net} = \sum_{k=1}^{P}\vec{F}_k + \sum_{j\neq i}\vec{F}_{ji} = \sum_{k=1}^{P}\vec{F}_k = \sum_{i=1}^{N} m_i \frac{d^2\vec{r}_i}{dt^2} = \frac{d^2}{dt^2}\sum_{i=1}^{N} m_i\vec{r}_i$ where we have used the fact that the internal forces cancel in pairs by Newton's 3rd law. If we define $\vec{R}_{cm} = \sum_i m_i\vec{r}_i / M$ where M is the total mass of the system then $\vec{F}_{net} = M\frac{d^2}{dt^2}\vec{R}_{cm}$.

Relative to the origin chosen, the total torque on the system about the CM is $\vec{\tau}_{net} = \sum_{i=1}^{P}(\vec{R}_i - \vec{R}_{cm})\times\vec{F}_i + \sum_{j\neq i}\vec{r}_i\times\vec{F}_{ji} = \sum_{i=1}^{P}\vec{R}_i\times\vec{F}_i - \vec{R}_{cm}\times\vec{F}_{net} + 0$ where we have used the fact that $\vec{r}_i\times\vec{F}_{ji} + \vec{r}_j\times\vec{F}_{ij} = \vec{r}_i\times\vec{F}_{ji} - \vec{r}_j\times\vec{F}_{ji} = \left(\vec{r}_i - \vec{r}_j\right)\times\vec{F}_{ji} = 0$ if the mutual forces are directed along the vector joining the particles. But then

$$\sum_{i=1}^{P}\vec{R}_i\times\vec{F}_i - \vec{R}_{cm}\times\vec{F}_{net} = \sum_i \frac{d\left(\vec{L}_i\right)_{cm}}{dt} = \sum_{i=1}^{N}(\vec{r}_i - \vec{R}_{cm})\times m_i\frac{d\vec{v}_i}{dt} = \frac{d\vec{\omega}}{dt}\sum_{i=1}^{N} m_i\left|\vec{r}_i - \vec{R}_{cm}\right|^2 = I_{cm}\vec{\alpha}.$$

where we have used the fact that for all the forces and force position vectors in the same plane as the CM, $\vec{v}_i = \left|\vec{r}_i - \vec{R}_{cm}\right| \frac{d\theta}{dt}\hat{\theta}$ and $\left(\vec{r}_i - \vec{R}_{cm}\right) \times \frac{d\theta}{dt}\hat{\theta} = \left|\vec{r}_i - \vec{R}_{cm}\right|\vec{\omega}$. Therefore

$$\vec{\alpha} = \frac{1}{I_{cm}}\left[\sum_{i=1}^{P} \vec{R}_i \times \vec{F}_i - \vec{R}_{cm} \times \vec{F}_{net}\right]$$

3) The ball will decrease its translational speed and increase its rotational speed until it rolls without slipping. Thereafter the ball will decelerate until it comes to rest.

The acceleration due to gravity acting along the plane is $-g\,sin\,\theta$ while that due to friction is $-\mu_k g\,cos\,\theta$. The translational acceleration of the ball while it is slipping is $-g\,sin\,\theta - \mu_k g\,cos\,\theta = -9.3ms^{-1}$. The tangential acceleration of the ball is $a_t = \dfrac{\mu_k mgR\,cos\,\theta}{\frac{2}{5}mR^2}R = 12.5ms^{-1}$. Therefore the translation and tangential speeds will be the same and the ball will rotate without slipped when $10 - 9.3t = 12.5t$ or at $t = 0.5\ s$ (to one significant figure). The ball will then roll without slipping. The ball at this point has a translational speed of 5.4 ms^{-1}, rolls without friction (no net slippage) and so decelerates at $-g\,sin\,\theta$=-5 ms^{-2}. The distance it travels in the slipping part of the trip is $\dfrac{v^2 - v_i^2}{2a} = \dfrac{100 - 5.4^2}{2(-9.3)} = 3.8m$. In the second part of the trip where it travels without slipping $\dfrac{v_f^2 - v^2}{2a'} = \dfrac{-5.4^2}{2(-5)} = 2.9m$. Therefore it travels a total distance of 6.7 m (7 m to one significant digit) up the incline.

ii) It rolls without slipping going down the incline (see iii) and if v_b is the speed at the bottom, we have $\frac{1}{2}mv_b^2 + \frac{1}{2}I\omega^2 = \frac{7}{10}mv_b^2 = mgh_{\max} = 5(10)(6.7)(\frac{1}{2}) = 170J$. The fraction of translation kinetic energy is $\dfrac{5/2}{1+5/2} = 5/7$ or 120 J. Therefore it has lost 250-120= 130 J of translational kinetic energy.

iii) If the ball slips the torque delivered by static friction is not sufficient to have the tangential acceleration equal the translational acceleration or $\dfrac{\mu_s g\cos\theta}{2/5} < g\sin\theta$.

Hence $\frac{5}{2}\cos\theta < \sin\theta$ or $\tan\theta > 5/2$. Hence θ >68^0.

4) The mass will continue to fall until the gain in potential energy of the spring is equal to the loss in potential energy from gravity. This will correspond to 2A.

i) Hence $\frac{1}{2}k(2A)^2 = 2mgA$ or $A = mg/k$ = 0.1 m

ii) The equilibrium position is obviously 0.1m below the starting point and so relative to this point the position is given by $x = 0.1cos(\Omega t + \phi)$ where $\Omega = \sqrt{k/m} = \quad = 10\,s^{-1}$. Given the position of $x = 0$ at $t=0.5$, we have $\Omega(0.5) + \phi = \pi/2$ or $\phi = \pi/2 - 5$. The velocity at t=0.75s is $v = -A\Omega\sin(\Omega t + \phi) = 1.\sin(2.5 + \pi/2) = 0.71$ ms^{-1}.

iii) If the end of the spring is displaced from equilibrium by an amount x and is moving with speed $\frac{dx}{dt}$, we have that any element of mass dm which is located at position y along the length is moving with speed $\frac{y}{L}\frac{dx}{dt}$. Therefore the kinetic energy of the spring is $\frac{1}{2}\int_0^L \rho\left(\frac{y}{L}\frac{dx}{dt}\right)^2 dy = \frac{1}{2}\frac{1}{3}m\left(\frac{dx}{dt}\right)^2$ where ρ is the density. Hence the system oscillates as if m/3 were attached to a massless spring.

5) Call F_t the frictional force exerted by each roller backward on the plank. Name as F_b the rolling friction exerted backward by the ground on each roller. Suppose the rollers are equally far from the ends of the plank. For the plank, $\sum F_x = ma_x$ or 6N-2F_t = 6kg(a_p). The centre of each roller moves forward only half as far as the plank since the velocity at the top of the roller is twice the velocity of the centre relative to the ground. Each roller therefore has acceleration $a_p/2$ and angular acceleration

$$\alpha = \frac{a_p/2}{.05m} = \frac{a_p}{0.1m}$$

Then for each roller $\sum F_x = ma_x$ or $F_t - F_b = 2kg(a_p/2)$ (1)

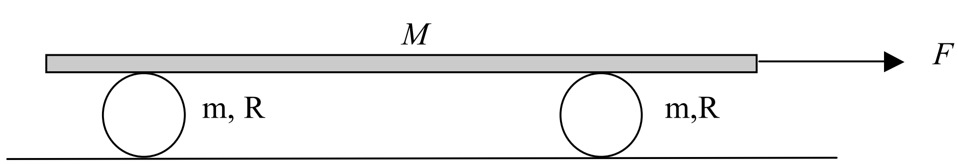
and $\sum \tau = I\alpha$ or $F_t(.05m) + F_b(.05m) = \frac{1}{2}(2kg)(.05)^2\frac{a_p}{.1m}$

so $F_t + F_b = (1/2kg)a_p.$ (2)

Adding (1) and (2) gives $2F_t = (1.5kg)a_p$ (3)

i) Therefore 6N-(1.5kg)a_p=(6N)a_p giving $a_p=0.8ms^{-2}$ and for each roller a = $a_p/2$=0.4ms^{-2}.

ii) Substituting into (3) gives $F_t = (1.5kg)a_p/2 = 0.6N$ and from (2) we have F_b=-0.2N, meaning that the bottom frictional force acts backwards.



6) A comet of mass 1.2x10^{10} kg moves in an elliptical orbit around the Sun. Its distance from the Sun ranges between 0.5 AU and 50 AU.

i) For an elliptical orbit of the form $r = \dfrac{l}{1 - e\cos\theta}$ the maximum distance is $\dfrac{l}{1-e} = 50AU$ and the minimum distance is $\dfrac{l}{1+e} = 0.5AU$. Dividing the two equations and solving for e one gets $e = 0.98$

ii) For a circular orbit one has that the speed is $v = \dfrac{2\pi R}{T}$. But we also have $\dfrac{mv^2}{R} = \dfrac{GM_s m}{R^2}$. This gives $T^2 = \dfrac{4\pi^2}{GM_s}R^3$. However, as explained in the book, for an elliptical orbit one has $T^2 = \dfrac{4\pi^2}{GM_s}a^3$ where a is the semimajor axis. 2a =max +min distance=50.5 AU so that a = 25.3 AU. Substituting one has T = 127 yrs. One could also simply remember that $T^2 = KR^3$ and use the fact that for the Earth T= 1 yr for a= 1AU to give K =1yr^2 /(1AU)^3 . One can then apply this relation to the comet.

iii) One simply has $U = -\dfrac{GM_s m}{r}$. Substituting the various numbers gives

U = -2.1x10^{17}J.