Solutions...........Final Exam..............2007

1. i) A Galilean transformation is one which relates kinematic quantities (such as position, velocity, acceleration) in one frame to those in another frame of reference which is moving with constant relative velocity.

ii) Since moment of inertia depends on axis of rotation, which can be any axis, the moment of inertia, which depends on the square of the distance of mass about that axis cannot be unique.

iii) In order to guarantee that the ball will roll without slipping it must be thrown in such a way that the tangential velocity of the bottom of the ball relative to the center is opposite in direction but with the same magnitude as the forward translational velocity of the ball.

iv) A collision is an interaction (not requiring contact) between two objects such that the momentum of the system is constant but the individual momenta might change.

v) The field of a physical parameter is the spatially dependent value of that parameter. A force field is a vector field giving the value of the force vector at any point in space.

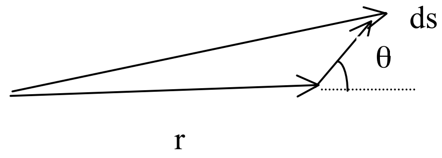
2. Let's call $\vec{r} = (1+t^2)\hat{i} + (-4t+3)\hat{j} + (1-t^2)\hat{k} = x_B\hat{i} + y_B\hat{j} + z_B\hat{k}$. From this one has that the velocity of the bee is $\vec{V}_B = 2t\hat{i} - 4\hat{j} - 2t\hat{k}$.

i) The bee lands when $x_B + y_B + 10z_B = 5$. This leads to the quadratic equation $9t^2 + 4t - 9 = 0$ which has solution $t_{1,2} = \dfrac{-2 \pm \sqrt{85}}{9} = \{-1.25, 0.8\}$. The only positive root is 0.8s.

ii) The normal to the surface is the vector $\vec{N} = (1,1,10)$. Hence the velocity is perpendicular if $\vec{N} \times \vec{V} = \vec{0}$. This gives $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 10 \\ 2t & -4 & -2t \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$. But this requires $-2t + 40 = 0$, $22t=0$, $-20t-4=0$, which do not yield a unique value of t; hence the velocity is never perpendicular to the table.

iii) The shadow of the fly has the same x and y co-ordinates as the bee so that the shadow is given by $\vec{r}_s = (1+t^2)\hat{i} + (-4t+3)\hat{j} + z_s\hat{k}$ where $x_B + y_B + 10z_s = 5$. This gives $z_s = \dfrac{1}{10}(-t^2 + 4t + 1)$. Hence $\vec{r}_s = (1+t^2)\hat{i} + (-4t+3)\hat{j} + \dfrac{1}{10}(-t^2 + 4t + 1)\hat{k}$ so that the velocity of the shadow is $\vec{v}_s = 2t\hat{i} - 4\hat{j} + (-t/5 + 2)\hat{k}$ whose magnitude is $v_s = \dfrac{1}{5}\sqrt{101t^2 - 4t + 404}$.

3. i) Since the gravitational force acts along the line of centers joining the planet and the sun, the torque or $\vec{\tau} = \vec{r} \times \vec{F} = \vec{0}$. Hence $\dfrac{d\vec{L}}{dt} = \vec{\tau} = \vec{0}$, *i.e.,* the angular momentum is



constant. But the angular momentum associated with the orbital motion is

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m(\vec{r} \times \frac{d\vec{r}}{dt}) = m\left(r\frac{ds}{dt}\sin\theta\right) == 2m(\frac{1}{2}\frac{r(ds)\sin\theta}{dt}) = 2m\frac{dA}{dt}$$

where s is the distance along the path of orbit and A is the area of the enclosed triangle (area swept out by radius vector when planet moves). Hence $\frac{dA}{dt} = 0$.

ii) If the planet is in a circular orbit then $\frac{mv^2}{R} = \frac{GM_s m}{R^2}$. Also $T = \frac{2\pi R}{v}$. Combining these two equations gives $\frac{T^2}{R^3} = \frac{4\pi^2}{GM_s}$, a constant.

iii) If $\vec{p} \times \vec{L} - Gm^2 M\hat{r}$ is a constant then its derivative with respect to time is zero. Taking such a derivative we find

$$\frac{d}{dt}(\vec{p} \times \vec{L} - Gm^2 M\hat{r}) = \frac{d\vec{p}}{dt} \times \vec{L} + \vec{p} \times \frac{d}{dt}\vec{L} - Gm^2 M\frac{d\hat{r}}{dt}.$$

Since the gravitational force is central (i.e., acts along the radius vector) there is no torque ($\vec{r} \times \vec{F}_g = 0$) and $\frac{d\vec{L}}{dt} = 0$. The 2[nd] term on the right hand side (RHS) vanishes. Also $\frac{d\vec{p}}{dt} = \vec{F}_g = -\frac{GMm}{r^2}\hat{r}$ since the gravitational force is the only force on the object. Using the definition of angular momentum ($\vec{L} = mr^2\frac{d\theta}{dt}\hat{z}$), we find that the 1[st] term on the RHS is $-GMm^2\frac{d\theta}{dt}(\hat{r} \times \hat{z}) = -GMm^2\frac{d\theta}{dt}(-\hat{\theta}) = GMm^2\frac{d\theta}{dt}\hat{\theta}$. Finally using the fact that $\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta}$ (as we showed earlier in the year) the 3[rd] term on the RHS is $-GMm^2\frac{d\theta}{dt}\hat{\theta}$. Collecting all three terms, $\frac{d}{dt}(\vec{p} \times \vec{L} - Gm^2 M\hat{r}) = 0$ and $\vec{p} \times \vec{L} - Gm^2 M\hat{r}$ is a constant (an interesting result of having a central force).

4. (i) The average force $\underline{F} = \Delta P/\Delta t$. During a duration of $\Delta t = 60$ s, for the bullets $\Delta P =$ 240 bullets* 0.025 (kg/bullet) * 1000 (m/s). = 6 * 1000 Ns. Hence, $\boldsymbol{\underline{F} = 100\ N}$.

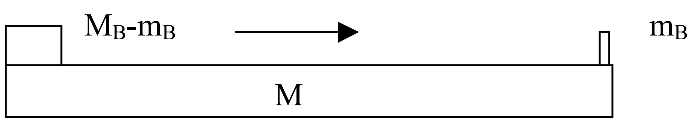
(ii) There are several ways to make the estimate. As long as one can state one's assumptions, a case can be made. I will consider what happens to a single bullet. The momentum gain ΔP_1 for one bullet is:

ΔP_1 = (0.025 kg)*1000 m/s = 25 (Ns) and the average force experienced by the single bullet in the gun is

F = $\Delta P/\Delta t$ = (25 Ns) / (2 * 10^{-3}) = **1.3 * 10^4 N.**

The real force is a complicated function of time. It likely peaks early and decays over 2 ms while the bullet is in the gun. If we assume a triangular shape for the force as a function of time the peak force is approximately twice the average force on a single bullet or about **3 x10^4 N**. By Newton's third law this is the same as the peak force on the gun.

(iii) Because the entire system is frictionless, there is not net external force on it and the center of mass of the system remains stationary. However, as bullets move (are fired) from one end towards the other end, mass is transferred and the rest of the system (flatcar plus machine gun, etc.) must move to keep the center of mass fixed.



At any time let the total mass of all the bullets be M_B and the mass of bullets at the left of the flatcar be m_B. Let the rest of the system have mass M so that the total mass of the system $M+M_B = 10^4$ kg. If the length of the car is L the center of mass of the system is given by

$$x_{cm} = \frac{1}{M + M_B}\left[Mx_{car} + (M_B - m_B)(x_{car} - L/2) + m_B(x_{car} + L/2)\right]$$

where x_{car} is the center of mass of the flatcar. Taking a time derivative, we have

$$\frac{dx_{cm}}{dt} = 0 = v_{car} + \frac{L}{M + M_B}\frac{d}{dt}m_B \text{ or } v_{car} = -\frac{L}{M + m}\frac{dm_B}{dt} =$$

$$-\frac{10m}{10^3 kg}(240/\min)(.025kg) = \mathbf{-10^{-3}\ m/s}$$

The minus sign means the car moves in the direction of the gun.

5.

i) As the force P increases from zero one would be inducing a torque which would have a tendency of causing the cylinder to rotate clockwise. The forces of static friction on the floor and the wall will prevent this from happening until they are limited by their maximum values, determined by the coefficient of static friction. Once the applied force becomes so large that the frictional toques' maximum values can no longer balance it, the object can rotate under the applied force with a net toque dictated by the coefficient of kinetic friction.

ii) Let's take the positive x-axis towards the right and the positive y-axis upwards. Identifying all the forces acting on the cylinder , we have that the forces acting along y are the force P, the weight $-Mg$, the normal force from floor, N_y , and the force of friction from wall, f_y. Hence for equilibrium we have:

$$P - Mg + N_y + f_y = 0. \qquad (1)$$

Similarly for the forces along x we have the force of friction at the bottom of the cylinder, f_x, and the normal force from the wall acting on the cylinder, N_x. Hence, for equilibrium we have:

$$N_x + f_x = 0. \qquad (2)$$

Using the definition of the coefficient of static friction, μ_s, we have that <u>the maximum forces </u>that static friction can supply (just before cylinder slips) are $f_x = \mu_s N_y$ and $f_y = -\mu_s N_x$. Eq. 1 then becomes

$$P - Mg + N_y - \mu_s N_x = 0 \qquad (1')$$

and Eq. 2 becomes:

$$N_x + \mu_s N_y = 0 \qquad (2')$$

Combining these two equations we have:

$$P - Mg + N_y[1 + (\mu_s)^2] = 0 \qquad (3)$$

For rotational equilibrium we have that the torque about the center of the cylinder is zero, hence

$f_y R + f_x R - PR = 0$ or $-\mu_s N_x + \mu_s N_y - P = (\mu_s)^2 N_y + \mu_s N_y - P = 0$. (4).

Solving 3 and 4 we have: $P = Mg\left[\dfrac{\mu_s(1+\mu_s)}{1 + 2\mu_s^2 + \mu_s}\right] = \dfrac{3}{8}Mg$.

6. Consider rotation about the pivot point. We have $\dfrac{dL}{dt} = I_p \dfrac{d^2\theta}{dt^2} = \tau = -mg\sin\theta$. Using the parallel axis theorem, $I_p = I_{CM} + md^2$, and the small angle approximation, $\sin\theta \approx \theta$, we have $\dfrac{d^2\theta}{dt^2} = \tau = -\dfrac{mgd}{I_{CM} + md^2}\theta$.

i) The angular frequency of the simple harmonic motion is $\Omega = \sqrt{\dfrac{mgd}{I_{CM} + md^2}}$ so that

$T = 2\pi\sqrt{\dfrac{I_{CM} + md^2}{mgd}}$.

ii) For the minimum period we require

$\dfrac{dT}{dd} = 0 = 2\pi\left(\dfrac{I_{CM} + md^2}{mgd}\right)^{-1/2}\left[\left(\dfrac{2md}{mgd}\right) + (-1)\left(\dfrac{I_{CM} + md^2}{mgd^2}\right)\right]$ or, since the term in square brackets is always positive, $2md^2 - (I_{CM} + md^2) = 0$. Hence the minimum period occurs for $I_{CM} = md^2$.