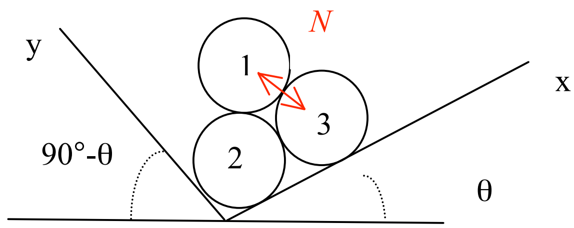
Problem Set 9 Solutions

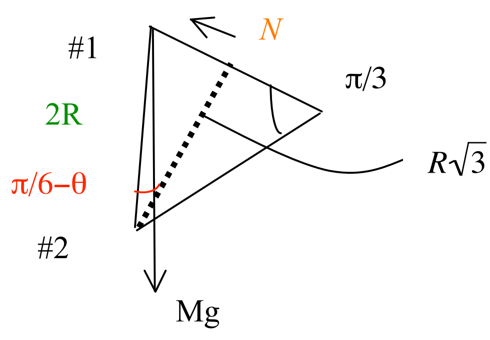
1.



Since this problem doesn't ask for values of all the unknowns, judicious choices for applying equilibrium conditions will simply the problem considerably. Consider the walls holding the cylinders to define x and y axes as shown. Number the cylinders as shown. The lines between centres make an equilateral triangle with interior angles $\pi/3$. If the cylinders are stacked as above, when the angle θ is reduced to its minimum value #3 *loses contact with #2,and begins to move, with no force between #2 and #3*. The total force on #3 along *x* comes only from the component of the normal force from the top cylinder, *N*, and the component of #3's weight along x:

$$F_x^{(3)} = -Mg\sin\theta + N\cos(\pi/3) = -Mg\sin\theta + N/2 = 0$$

Now we just need another condition which only involves *N*. But in equilibrium the torque on cylinder #1 about any point is zero. Let's evaluate the torque about the center of #2. This torque results from the weight of #1 acting through the center of #2 and which, acting vertically downward, makes an angle of π/6–θ with respect to the position vector joining #1 and # 2. In addition there is the torque from a normal force, *N,* from #3 directed along the line centers of #2 and #3 and acting at the midway point at a perpendicular distance $R\sqrt{3}$ from the center of circle #2. There is no torque from the normal force between #1 and #2 since it acts parallel to the vector connecting #1 and #2.



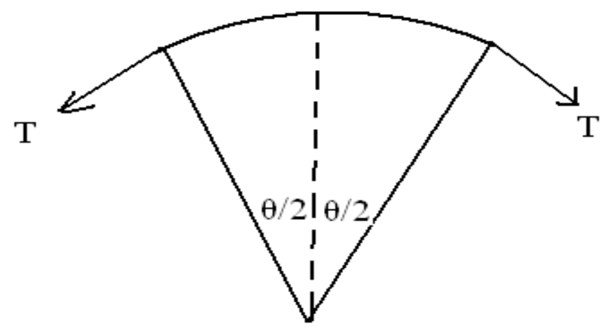
$$\tau_z^{(1)} = -(2R)Mg\sin(\pi/6-\theta) + N(R\sqrt{3}) = 0\,.$$

Eliminating N between the two equations gives $-\sin(\pi/6-\theta)+\sqrt{3}\sin\theta = 0$. Expanding the sine function on the left and dividing by $\cos\theta$ gives $\tan\theta = \dfrac{1}{3\sqrt{3}}$ or $\theta = 11^0$.

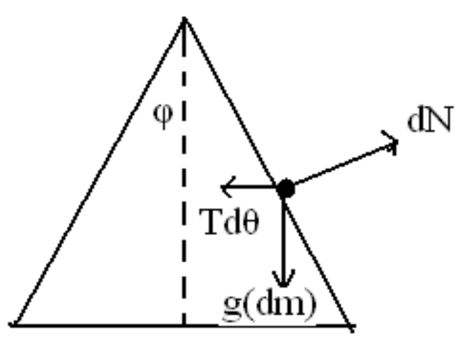
2. Let the radius of the cone where the chain sits be R and the vertical position above the base be H. Suppose the chain is displaced a small distance vertically. To do this it would have to increase its perimeter to $2\pi(R+\Delta R)$, in which case it now sits at a height $H+\Delta H$, where $2\pi\Delta R = -2\pi\frac{r}{h}\Delta H$. To expand the change the chain would have to do work $T(2\pi)\Delta R$ and this would have to be provided by the change in gravitational potential energy $-W\Delta H$. Equating the two energy changes while using the relation between ΔH and ΔR gives $T = W/(2\pi)\frac{h}{r}$.

Alternative approach

Consider a small piece of the chain subtended by a small angle θ on the cone (eventually we will take θ to be infinitesimally small), from the top view:



This piece of chain has two tension forces, one on either end, whose resultant net force is acting inwards of magnitude $F_{tension} = 2T\sin(\frac{\theta}{2})$. If θ is small sin(θ/2) ≈ θ/2. Hence, for an infinitesmal angle, the net tension force is $dF_{tension} = Td\theta$. If we assume the mass of the chain m is evenly distributed, then a small piece of chain subtended by angle dθ has mass $dm = m\frac{d\theta}{2\pi}$. Now consider this small piece of chain from the side view of the cone:



The small piece of chain has three forces acting on it: the normal force, the tension force, and the gravitational force. Hence for equilibrium or horizontal and vertical forces:

$Td\theta - \cos(\varphi)dN = 0$

$- gdm - \sin(\varphi)dN = 0$ or $\frac{T}{g}\frac{d\theta}{dm} = \cot(\theta)$ so $T = g\frac{m}{2\pi}\cot(\varphi)$.

Since W=mg, and cot(φ) = h/r, $T = \frac{W}{2\pi}\frac{h}{r}$.

3. The space shuttle moves from a circular orbit of radius R = 250 km to an elliptical orbit, with max distance 325 km and min distance 200 km above surface of earth. In general the orbit of the satellite is given by

$r = \frac{l}{1 - e\cos\theta}$ with the center of the earth will located at a focus. The minimum distance (perigee) and the maximum distance (apogee) are $\frac{l}{1 \pm e}$ or a(1-e) and a(1+e) respectively, with a being the semi-major axis (=$\frac{1}{2}\left[\frac{l}{1+e} + \frac{l}{1-e}\right]$).

We use G = $6.67300 * 10^{-11} m^3 / kg / s^2$; M= mass of earth = $5.9742 * 10^{24} kg$

R= mean radius of earth= $6.378 \ * 10^3 km$; m = mass of space shuttle = $2 * 10^4 kg$

r = distance above earth in circular orbit= 250 km

Xmax = max distance from surface of earth = 325 km

Xmin = min distance from surface of earth = 200 km

a =R+ $(X_{max} + X_{min})/2$

The total energy of the satellite in orbit is (see *Serway* or notes from class for proof):

$E = -\frac{GMm}{2a}$

In circular orbit $E_i = -\frac{GMm}{2(R + r)} = -6.015 * 10^{11} J$

In elliptical orbit $E_f = -G\frac{Mn}{2a}$ = $-6.003 * 10^{11} J$

Hence, the change in total energy is $E_f - E_i = 1.2 * 10^9 J$.

When in the circular orbit, the kinetic energy *K* is the total energy minus the potential energy U, or $K = \frac{GMm}{2(R + r)}$. But $K = P^2 / 2m$, where P is the magnitude of the linear momentum. Hence $P = m\sqrt{\frac{GM}{R + r}}$ and the initial angular momentum in circular orbit

$L_i = (R + r)P = m\sqrt{GM(R + r)} = 1.028 * \ 10^{15} kgm^2 / s$.

Similarly, when in an elliptical orbit, at the maximum distance the kinetic energy of the satellite is $K_{\max} = -\dfrac{GMm}{2a} + \dfrac{GMm}{R + X_{\max}}$. Hence the linear momentum $P_{\max} = \sqrt{2mK_{\max}}$, and the (constant) angular momentum

$$L_f = (R + X_{\max})P = X_{\max}\sqrt{2\frac{GMm^2}{R + X_{\max}} - G\frac{Mm^2}{a}}$$

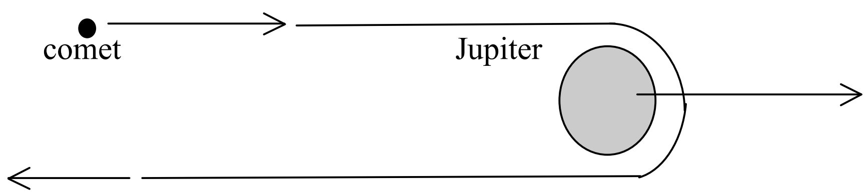
or, after a little algebra $L_f = m\sqrt{\dfrac{GM}{a}((R + X_{\max})(R + X_{\min}))}$

Note that this expression is symmetric in X_{max} and X_{min}. Hence

$L_f = 1.029 * 10^{15} kgm^2/s$ and $\Delta L = L_f - L_i = 1*10^{12} kgm^2/s$.

Note that angular momentum is a vector; in this case the direction is orthogonal to the plane of the motion of the earth-space shuttle dynamics.

4. This is almost a research problem since it doesn't ask for a closed-form solution but rather some creative thinking to imagine a scenario in which the comet can be captured. While one could solve the problem exactly, our purpose here is to only show how Jupiter can help to capture a comet under some condition. If the comet starts off with negligible kinetic energy at a distance far away from the sun or Jupiter then it initially has total energy equal to zero. If it is to eventually be bound to the sun, then it must lose some of energy by giving it to Jupiter so its total energy will be negative . When the comet gets to a position where its radial distance from the sun is the same as Jupiter's (R) but it is >> R_J away from Jupiter, where R_J is Jupiter's planetary radius (this might correspond to a position far behind Jupiter in it's orbit, so the comet initially doesn't feel Jupiter's pull), and it would have a speed equal to its escape speed from the sun at that position or $V_{escape} = \sqrt{2\dfrac{GM_{sun}}{R}}$. In its (nearly circular) orbit Jupiter is initially traveling at $V_J = \sqrt{\dfrac{GM_{sun}}{R}} = \dfrac{1}{\sqrt{2}}V_{esc}$. Now let's consider a "collision" .



Since no energy is dissipated through friction, etc. we can assume conservation of kinetic energy. Using the equations for 1-D collisions in Ch. 9 (Eq. 9.21) and assuming the mass of the comet is much less than that of Jupiter, for the comet's final velocity long after the collision $V_f^c = -V_i^c + 2V_J = -0.4V_{esc}$. Obviously, in 2-D there are other possibilities that allow the comet to have less than it's escape speed after the interaction. Note: in all cases Jupiter's velocity changes only slightly.