Solutions: set # 6

#1 a)

Method 1)

The initial and final positions are $\vec{r}_i$ = (1,1) and $\vec{r}_f$ $\overline{r_f}$=(20,y). The gecko accelerates according to $\vec{a} = \frac{F}{M} = \frac{(0.05N,-0.06N)}{0.04kg} = (1.25,-1.5)\frac{m}{s^2}$. Now, since the initial velocity is 0, use $\Delta x = \frac{1}{2}a_x t^2, \Delta y = \frac{1}{2}a_y t^2$. Therefore

(1) $19m = \frac{1}{2} * 1.25\frac{m}{s^2} * t^2$

(2) $y - 1 = \frac{1}{2} * -1.5\frac{m}{s^2} * t^2$. This gives y=-2.x10[1] m (to one significant figure).

Method 2)

Since the mass starts at rest, the total work done (W) is equal to the final energy:

$\vec{F}\circ\overline{\Delta \mathrm{r}} = \frac{1}{2}mv^2$ $\vec{F}\cdot(\Delta\vec{r}) = \frac{1}{2}mv^2$ or $0.05N * 19m + (-0.06N) * (y - 1m) = \frac{1}{2} * 0.04kg * (11\frac{m}{s})^2$.

This gives y=-2.x10[1]m.

#1 b)

The force, $\vec{F}(\vec{r})$ depends only on the magnitude of $\vec{r}$, and is in the direction of $\vec{r}$, so $\vec{F}(\vec{r}) = F(r)\hat{r}$

The work done by such a force along a 3D path is: $\int_{\vec{r}_1}^{\vec{r}_2} F(\vec{r})\cdot d\vec{r} = \int_{r_1}^{r_2} F(r)dr$ $\int_{r_1}^{r_2}\vec{F}(\vec{r})\circ\overline{dl} = \int_{r_1}^{r_2} F(r)dr$.

The dot product picks out only the radial component of $d\vec{r}$ because all non-radial components of $\vec{F}$ are 0. Because *F(r)* is only a function of a single variable, it can be written as the (negative by conventional definition of potential energy) derivative of some other function $F(r) = \frac{-dU(r)}{dr}$. So,

$$W = \int_{r_1}^{r_2} F(r)dr = \int_{r_1}^{r_2}\frac{-dU(r)}{dr}dr = U(r_1) - U(r_2) = U(\vec{r}_1) - U(\vec{r}_2)$$

Obviously this does not depend on the path taken, only on the end-points; hence, the force is a conservative force.

#2

At the top of the hemisphere, the total energy (which remains constant) for the object at rest is potential energy = $E=mgR$, taking the zero of potential energy to be zero at the base of the hemisphere. If θ is defined to be the angle from the x-direction to the mass position (*i.e.* θ = π/2 at the top, and as the bead drops, θ decreases toward 0), then its total energy is $E = mgR\sin\theta + 1/2mv^2 = mgR$. So $v^2 = 2gR(1 - \sin\theta)$.

The acceleration experienced by the bead may be decomposed into tangential and radial components. The radial component of the acceleration is $a_r = \frac{-v^2}{r}$=-2g(1-sinθ) (negative because it's accelerating toward the origin). The forces acting in the radial direction are the component of gravity (-mg*sin(θ)) and the normal force. **The block will lose contact with the surface when the normal force is 0** or -mg*sinθ)=-2g(1-sinθ) or sinθ=2/3. Once the bead loses contact with the hemisphere it moves according to projectile motion. If the center of the hemisphere is the origin, the bead's initial coordinates are (Rcosθ, Rsinθ)=(2R/3,$R\sqrt{\frac{5}{9}}$). The initial velocity vector is

(v*sin θ, -v*cosθ)=($\frac{2}{3}\sqrt{\frac{2}{3}gR}$, $\sqrt{\frac{5}{9}}\sqrt{\frac{2}{3}gR}$) = (0.76, -0.85) m/s. The direction comes from the fact that the velocity is at 90^0 to the position vector. Then $\Delta y = v_{y0}t + 1/2 a_y t^2$. This gives t=0.1 s

$\Delta x = v_x t = 0.08$ m

We want the distance from the edge of the hemisphere (call that D):

$D = x_0 + \Delta x - R =$ 0.03 m

Alternate method of finding θ:

Suppose the bead was fixed to the hemisphere so it could slide but couldn't come off, and slid to the right. Then it would start by slowly accelerating to the right, and downward. But as it approaches the table, the horizontal component of its velocity would drop to 0 because it would hit while moving straight down (the angle between the ground and the edge of the hemisphere is 90^0). So at some point it would have to accelerate to the left. For this question, the bead is not fixed to the hemisphere, so there's no force that can push it to the left, so it cannot accelerate to the left, so **it loses contact when the horizontal component of its acceleration is 0.**

This occurs when v_x=v*sin(θ)=$\sqrt{2gR(1-\sin(\theta))} * \sin(\theta)$ is at a maximum (Use sin rather than cos here because the velocity is at 90^0 to the position vector). But if v_x is at a maximum, then so must $(v_x)^2$, which is a lot easier to work with. $(v_x)^2 = 2gR(1-\sin(\theta))*\sin^2(\theta) = 2gR(\sin^2(\theta) - \sin^3(\theta))$.

$$\frac{d(v_x^2)}{d\theta} = 2gR\frac{\partial}{\partial\theta}\left(sin^2(\theta) - \sin^3(\theta)\right) = 2gR\left(2\sin(\theta)\cos(\theta) - 3\sin^2(\theta)\cos(\theta)\right) = 0$$

For which the only solution is sin(θ)=2/3. This method is physically the same as the previous method because it is the normal force that gives horizontal acceleration.

#3

Given: U(r)=$U_0(\left(\frac{\sigma}{r}\right)^4 - \left(\frac{\sigma}{r}\right)^2)$ the equilibrium position is where U(r) is a minimum or

$\frac{d(U(r))}{dr} = U_0\left(\frac{-4}{\sigma}\left(\frac{\sigma}{r}\right)^5 + \frac{2}{\sigma}\left(\frac{\sigma}{r}\right)^3\right) = 0$, so $\frac{2}{\sigma}\left(\frac{\sigma}{r}\right)^3 = \frac{4}{\sigma}\left(\frac{\sigma}{r}\right)^5$. This gives $r_0 = \sqrt{2}\,\sigma = 1.4\,\sigma$ where r_0 is the equilibrium position. One can verify by taking the second derivative of *U(r)* evaluated at r_0 (the second derivative is >0) that this position corresponds to a stable equilibrium.

$U(r_0) = U_0\left(\left(\frac{\sigma}{r_0}\right)^4 - \left(\frac{\sigma}{r_0}\right)^2\right) = -\frac{1}{4}U_0$

Given the speed of the hydrogen at the equilibrium position is $v_0 = 800$ m/s, the total energy (which is conserved) is $E = \frac{1}{2}mv_0^2 - \frac{1}{4}U_0$. For a hydrogen atom, $m = 1.7\times10^{-27}$kg.

The maximum distance occurs when the energy is entirely potential energy, *i.e.* H is not moving. Therefore

$E = \frac{1}{2}mv_0^2 - \frac{1}{4}U_0 = U_0\left(\left(\frac{\sigma}{r}\right)^4 - \left(\frac{\sigma}{r}\right)^2\right)$. This is a quadratic equation in $\left(\frac{\sigma}{r}\right)^2$ giving 2 solutions:

$\left(\frac{\sigma}{r}\right)^2 = 0.34$ or r = 1.7 σ, and $\left(\frac{\sigma}{r}\right)^2 = 0.66$ which gives r = 1.2 σ.

Note that these are on opposite sides of the equilibrium position. The maximum displacement from equilibrium on either side is 1.7σ-1.4σ=0.3 σ or 1.41σ-1.23σ=0.2σ. So the first solution gives the largest displacement with 0.3 σ = $9.\ \times10^{-11}$m = 0.9 picometers.

#4

The linear density (mass per unit length) of the chain is $\rho = \frac{M}{L} = 1\ \frac{kg}{m}$. If the spring is moving, then the velocity of some part of the spring (position x) depends on x: If the end is moving at a velocity $v(L) = v_0$, then $v(0)=0$ (one end is stationary), and $v(x) = v_0\frac{x}{L}$. The differential kinetic energy (dE_k) of a small piece of spring at position x (length dx, mass $dm = \rho dx$) is

$dE_k = d(\frac{1}{2}mv(x)^2) = (\frac{1}{2}\frac{v_0^2x^2}{L^2})(dm) = \frac{\rho v_0^2x^2}{2L^2}dx$ and the total kinetic energy, E_K is:

$E_k = \int dE_k = \int_0^L \frac{\rho v_0^2 x^2}{2L^2}dx = \frac{1}{6}v_0^2M$

If the spring is displaced by x_0 initially, then $E = \frac{1}{2}kx_0^2$ with no kinetic energy. The total energy remains constant, so when the energy is entirely kinetic:

$E = \frac{1}{2}kx_0^2 = \frac{1}{6}v_0^2M$

Solving for v_0: $v_0 = \sqrt{\frac{3k}{M}}x_0$

If we assume a cosine-like oscillation $x(t) = x_0\cos(\omega t)$, then $v(t) = -x_0\omega\sin(\omega t)$. The maximum velocity is $x_0\omega$, and so $\omega = \sqrt{\frac{3k}{M}} = 8\ s^{-1}$. Note that this is only different from the usual mass-on-spring frequency by a small constant factor. Therefore a spring with uniformly distributed mass *M* behaves the same as a mass-less spring with a mass *M*/3 at the end….a useful result to introduce realism in many problems.