Problem Set #2 Solutions

1. We have $r\cos(\theta-\theta_0) = r\left[\cos\theta\cos\theta_0 + r\sin\theta\sin\theta_0\right] = r_0$. Defining $(x_0, y_0) = (r_0\cos\theta_0, r_0\sin\theta_0)$ and $(x, y) = (r\cos\theta, r\sin\theta)$, we have $r_0^2 = xx_0 + yy_0$. This is the equation for a line with y-intercept $y_{\text{int}} = \dfrac{r_0^2}{y_0} = \dfrac{r_0}{\cos\theta_0}$. Note that we can also re-write the equation as $r_0^2 = (x, y)\bullet(x_0, y_0)$, where $A\bullet B$ denotes dot product. Hence, the projection of the vector (x,y) onto the vector from the origin to the fixed point is of the length r_0, and hence the fixed point is the closest point to the origin. Alternatively, from the equation of the line in polar co-ordinates, the secant function has its minimum value (unity) when $\theta = \theta_0$. This gives the minimum value of $r = r_0$. Hence the minimum distance of the line from the origin occurs for the fixed point.

2. The motion in the (x,y) plane is a circle while that in the z-direction is translational. The path is usually called a "spiral" or a "helix". At t = 1.5s, $\vec{r} = (\cos(4.5), \sin(4.5), 13.5) = (-0.21, -0.98, 13.5)$. Hence $|\vec{r} - \vec{r}_0| = \sqrt{(\cos(4.5)+1)^2 + (\sin(4.5)-2)^2 + 13.5^2} = 14m$. From the 2nd derivative of the position vector, $\vec{a} = (-9\cos(3t), -9\sin(3t), 12) = (1.9, 8.8, 12)$. Note that $r(t=1.5) = 13.5$, and $a(t=1.5) = 15$ (mks units). If the angle between the position and acceleration vectors is θ, then $\cos\theta = \dfrac{\vec{a}\bullet\vec{r}}{ar} = \dfrac{1.5x10^2}{15*13.5} = 0.74$. Hence, θ=0.74 rad (numbers are an unintended coincidence!). The speed of the proton is given by $v = |\vec{v}(t)| = \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2 + (12t)^2} = \sqrt{9+144t^2}$. You can minimize this by setting the 1st derivative to zero, etc. or note by inspection that the minimum occurs for t = 0, and gives a minimum speed of 3 m/s.

3. Let the speed and the launch angle of the ball be v_0 and θ_0 respectively. Setting the origin at the launch point let the hill be given by $y = \tan(15^0)x$ and the trajectory of the ball until it hits the hill be $(x, y) = (v_0\cos(\theta_0)t,\ -\dfrac{g}{2}t^2 + v_0\sin(\theta_0)t)$ where time is measured from launch. If t_h is the landing time t_h corresponding to point (x_h, y_h) then

$y_h = v_0\cos(\theta_0)t_h\tan(15^0) = -\dfrac{g}{2}(t_h)^2 + v_0\sin(\theta_0)t_h$ or

$t_h = \dfrac{2v_0}{g}(\sin(\theta_0) - \cos(\theta_0)\tan(15^0))$ and $x_h = 2\dfrac{v_0^2}{g}(\sin(\theta_0)\cos(\theta_0) - (\cos(\theta_0))^2\tan(15^0))$.

x_h (and y_h) are maximal if $\dfrac{dx_h}{d\theta_0} = 2\dfrac{v^2}{g}(\cos(2\theta_0) + \sin(2\theta_0)\tan(15^0)) = 0$ or

$-\cot(15^o) = \tan(2\theta_0)$; $\theta_0 = 53^0$ (0.93 rad). Since t_h=4s, v_0 = 31 m/s. At t_h=3s, we have

$3 = \frac{2(31)}{g}(\sin(\theta_0) - \cos(\theta_0)\tan(15^0))$ or $\theta_0 = 42^0$ (0.73 rad). Hence,

$x_h = 31\cos(0.73)3 = 70m$. hence the distance up the incline is $\frac{x_h}{\cos(15^0)} = 72m$

4. Choose the x-axis along sea level and the y-axis to pass through the peak of the mountain. Let $(x_B, 0), x_B \le -1km$ be the location of ship B, and let θ_0 be the launch angle for the projectile. If v_0 is the muzzle speed, the trajectory is given by $(x, y) = (x_B + v_0\cos(\theta_0)t, -\frac{g}{2}t^2 + v_0\sin(\theta_0)t)$. In the absence of obstructions the horizontal range of the projectile is $R = v_0^2\sin(2\theta_0)/g$. However, the projectile must clear the peak. If t_p is the time it takes to reach the horizontal location of the peak, then $x_B + v_0 t_p\cos(\theta_0) = 0$. At that time, the height of the projectile is $y_P = -\frac{g}{2}\left(\frac{x_B}{v_0\cos\theta_0}\right)^2 - \tan(\theta_0)x_B \ge 1000$ or $\frac{g}{2}x_B^2 + \frac{v_0^2}{2}\sin(2\theta_0)x_B + 1000v_0^2\cos^2(\theta_0) \le 0.$

Or, substituting values for g and v_0 we have $5x_B^2 + \frac{v_0^2}{2}\sin(2\theta_0)x_B + 1000v_0^2\cos^2(\theta_0) \le 0.$. Since the maximum (unobstructed) range is 9 km (for $\theta_0 = \pi/4$), but this does not satisfy the above inequality (projectile doesn't clear peak) we must use a steeper angle. Larger launch angles reduce R, bringing the projectile closer to the (far) shore. The largest angles are obtained when ship B is closest to the shore or x_B = -1000 m. Hence for v_0 =*0.3 km/s* we have $1 - 9\sin(2\theta_0) + 18\cos^2(\theta_0) \le 0$. This gives (trying different values) $50^0 < \theta_0 < 86^0$ and $1.3km < R < 8.8km$ or, to a single significant digit, boat A must hug the shore to be safe.