PHY180F
Solutions Problem Set 9

1.

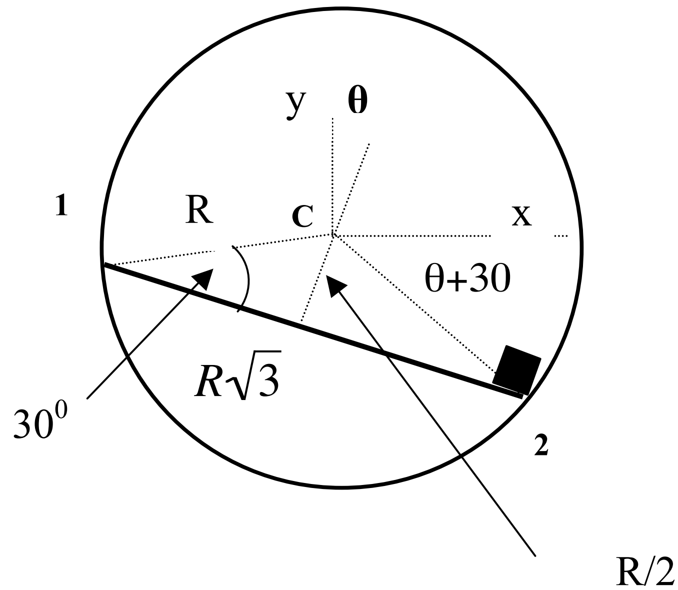

Fig. 1.1

Let $\vec{F} = F_x\hat{i} + F_y\hat{j}$ be the reaction force of the trough on the plank at point 1 and $\vec{G} = G_x\hat{i} + G_y\hat{j}$ the corresponding forces at point 2. On a frictionless surface the reaction force can act only normal to the surface, in this case towards the centre of the circle. From the diagram one then has $F_x = F\,cos(30-\theta)$, $F_y = F\,sin(30-\theta)$, $G_x = -G\,cos(30+\theta)$, $G_y = G\,sin(30+\theta)$ for translational equilibrium we have that the x, y components of the total forces must be zero or,

$$F_x + G_x = F\,cos(30-\theta) - G\,cos(30+\theta) = 0 \qquad F = G\frac{cos(30+\theta)}{cos(30-\theta)} \tag{1.1}$$

and

$$F_y + G_y - Mg - Mg/2 = F\,sin(30-\theta) + G\,sin(30+\theta) - 3Mg/2 = 0. \tag{1.2}$$

For rotational equilibrium about the point 1 we have

$$\tau_1 = -Mg\cos\theta + G\sin(30+\theta)\cos\theta - G\cos(30+\theta)\sin\theta = -Mg\cos\theta + G\sin(30) = 0$$

or $G = 2Mg\cos\theta$. (1.3)

Combining (1.1), (1.2) and (1.3) we have

$$G\frac{\cos(30+\theta)}{\cos(30-\theta)}\sin(30-\theta) + G\sin(30+\theta) = 3Mg/2 = \frac{3G}{4\cos\theta}$$

$$\cos\theta\big(\cos(30+\theta)\sin(30-\theta) + \cos(30-\theta)\sin(30+\theta)\big) = \frac{3}{4}\cos(30-\theta)$$

$$\cos\theta\big(\sin 60\big) = \frac{3}{4}\cos(30-\theta)$$ Hence $\theta = 30^0$.

2. The distance from the Earth's centre is obtained by equating the Gravitational force to the centripetal force:

$$\frac{GMm_{SAT}}{r^2} = m_{SAT}\omega^2 r, \qquad (2.1)$$

where r is the distance from the Earth's centre and ω is the angular frequency of the orbital motion. Then

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3}. \qquad (2.2)$$

If the satellite has the same orbital period as the period of the Earth (24 hours) then $\omega = 2\pi / T = 7.27\times10^{-5}\,s^{-1}$ and the height above the Earth is

$$r - R_E = \left(\frac{6.67\times10^{-11}\left(Nm^2/kg\right)\times 5.974\times10^{24}\,kg}{\left(7.27\times10^{-5}\,s^{-1}\right)^2}\right)^{1/3} - 6.38\times10^6\,m = 35.2\times10^6\,m$$

(2.3)

A circular geosynchronous orbit in the plane of the Earth's equator has a radius of approximately ***35,200 km***. Since the plane of the orbit must pass through the center of the Earth, it is not possible to establish a geosynchronous orbit at any other latitude than that of the equator. Of course many orbits with the same period (24hrs) are possible, but only the one can maintain latitude. This is somewhat of a moot point in practice since the satellite is so far away from the Earth that it is really "above" the entire face of one side of the Earth.

3. If $\vec{p}\times\vec{L}-Gm^2M\hat{r}$ is a constant then its derivative with respect to time is zero. Taking such a derivative we find

$$\frac{d}{dt}(\vec{p}\times\vec{L}-Gm^2M\hat{r})=\frac{d\vec{p}}{dt}\times\vec{L}+\vec{p}\times\frac{d}{dt}\vec{L}-Gm^2M\frac{d\hat{r}}{dt}.$$

Since the gravitational force is central (i.e., acts along the radius vector) there is no torque ($\vec{r}\times\vec{F}_g=0$) and $\frac{d\vec{L}}{dt}=0$. The 2nd term on the right hand side (RHS) vanishes. Also $\frac{d\vec{p}}{dt}=\vec{F}_g=-\frac{GMm}{r^2}\hat{r}$ since the gravitational force is the only force on the object. Using the definition of angular momentum ($\vec{L}=mr^2\frac{d\theta}{dt}\hat{z}$), we find that the 1st term on the RHS is $-GMm^2\frac{d\theta}{dt}(\hat{r}\times\hat{z})=-GMm^2\frac{d\theta}{dt}(-\hat{\theta})=GMm^2\frac{d\theta}{dt}\hat{\theta}$. Finally using the fact that $\frac{d\hat{r}}{dt}=\frac{d\theta}{dt}\hat{\theta}$ (as we showed earlier in the year) the 3rd term on the RHS is $-GMm^2\frac{d\theta}{dt}\hat{\theta}$. Collecting all three terms, $\frac{d}{dt}(\vec{p}\times\vec{L}-Gm^2M\hat{r})=0$ and $\vec{p}\times\vec{L}-Gm^2M\hat{r}$ is a constant (an interesting result of having a central force).

4. The shuttle is initially in a circular orbit with the centripetal force provided by gravity. If r_i is its initial distance from the Earth's center then:

$$\frac{m_s v_i^2}{r_i}=\frac{GM_E m_s}{r_i^2} \text{ or } \frac{1}{2}m_s v_i^2=\frac{1}{2}\frac{GM_E m_s}{r_i}. \qquad (4.1)$$

After the shuttle fires its rockets the orbit is no longer circular. The total energy is

$$E=\frac{1}{2}m_s(1.05v_i)^2-\frac{GM_E m_s}{r_i}=\frac{1}{2}m_s v^2-\frac{GM_E m_s}{r}, \qquad (4.2)$$

for any new speed, v, and distance r. The new angular momentum is $L=(1.05v_i)m_s r_i$ since the velocity is perpendicular to the radius vector. At the maximum (apogee)and minimum (perigee) distance the radius vector is perpendicular to the velocity vector for an elliptical orbit so that $L=(1.05v_i)m_s(R_E+h_i)=m_s v_m r_m$ or $v_m=1.05v_i r_i/r_m$. Inserting this speed into Eq. 4.2, using 4.1 and solving for r_m we have $1-\frac{r_i^2}{r_m^2}=\frac{2}{1.05^2}\left(1-\frac{r_i}{r_m}\right)$. This has the solution $r_m=r_i$ (minimum distance is the point where the rocket fired, as well as on opposite side of Earth) or $r_m=1.23r_i$ (maximum distance, corresponding to a height above the earth of ~1500 km. A little boost goes a long way!