**PHY180F**
**Solutions Problem Set # 8**

**1.** The clay ball adheres to the stick upon collision. The conservation of linear momentum principle leads to:

$$\mathbf{p}_i = \mathbf{p}_f$$
$$m_b v_0 = (m_b + M) v_{CM}\,. \tag{1.1}$$

Setting $m_b = 0.05\,\text{kg}$, $M = 0.3$ kg and $v_0 = 1\,\text{m/s}$, then $v_{CM} = 0.143\,\text{m/s}$. Choosing the center of the stick as the origin, the position of the center of mass (CM) along the horizontal axis is:

$$x_{CM} = \frac{m_b d}{m_b + M} = \frac{(0.05\text{kg}) \times (0.25\text{m})}{0.35\text{kg}} = 0.0357\text{m}\,, \tag{1.2}$$

where d = 25 cm from the center of the stick. The conservation of the angular momentum around the CM is expressed as:

$$L_i = L_f$$
$$(d - x_{CM}) m_b v_0 = I_{ball}\omega + I_{stick}\omega\,. \tag{1.3}$$

The moment of inertia of the stick around the CM of the system is found from the parallel axis theorem:

$$I_{stick} = I_{CM} + Mx_{CM}^2 = \frac{1}{12}ML^2 + Mx_{CM}^2 = \frac{1}{12}(0.3\text{kg})(1\text{m})^2 + (0.3\text{kg})(0.0357\text{m})^2\,,$$
$$I_{stick} = 0.0254\text{kg} \times \text{m}^2. \tag{1.4}$$

The moment of inertia of the ball about the CM is:

$$I_{ball} = m_b (d - x_{CM})^2 = 0.05\text{kg} \times (0.214)^2 = 0.0023\text{kg} \times \text{m}^2\,. \tag{1.5}$$

(i) By inserting the results (1.4) and (1.5) into (1.3), we obtain the angular speed after collision: $\omega = 0.39$ rad/s.

(ii) The kinetic energy of the system afterwards is given by:

$$K = K_{rot} + K_{CM} = \frac{1}{2}I\omega^2 + \frac{1}{2}(m_b + M)v_{CM}^2 = 0.0057J \tag{1.6}$$

**2.** For a single particle, the angular momentum vector is defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is radius vector and $\mathbf{p}$ is the momentum. For a system of particles (Fig. 2.1):

$$\mathbf{L} = \sum_j \mathbf{r}_j \times \mathbf{p}_j \, . \tag{2.1}$$

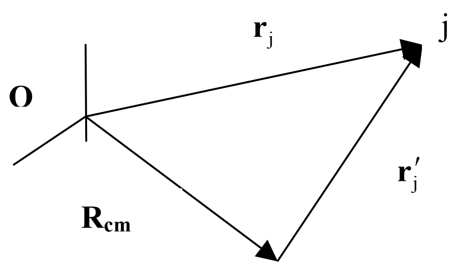


**Fig. 2.1**

We have the following relationships relative to the center-of-mass system of coordinates:

$$\mathbf{r}_j = \mathbf{R}_{CM} + \mathbf{r}'_j \, , \tag{2.2}$$

$$\mathbf{v}_j = \mathbf{v}_{CM} + \mathbf{v}'_j \, . \tag{2.3}$$

With $\mathbf{p}$ =m$\mathbf{v}$, the total angular momentum reads:

$$\mathbf{L} = \sum_j m_j \left(\mathbf{R}_{CM} + \mathbf{r}'_j\right) \times \left(\mathbf{v}_{CM} + \mathbf{v}'_j\right) . \tag{2.4}$$

Explicitly:

$$\mathbf{L} = \sum_j m_j \left(\mathbf{R}_{CM} \times \mathbf{v}_{CM} + \mathbf{R}_{CM} \times \mathbf{v}'_j + \mathbf{r}'_j \times \mathbf{v}_{CM} + \mathbf{r}'_j \times \mathbf{v}'_j\right). \tag{2.5}$$

$$\mathbf{L} = M\mathbf{R}_{CM} \times \mathbf{v}_{CM} + \mathbf{R}_{CM} \times \frac{d}{dt}\left(\sum_j m_j \mathbf{r}'_j\right) + \left(\sum_j m_j \mathbf{r}'_j\right) \times \mathbf{v}_{CM} + \sum_j m_j \mathbf{r}'_j \times \mathbf{v}'_j . \tag{2.6}$$

The relation $\sum_j m_j = M$ (M as the total mass of the system) has been introduced. Since $\sum_j m_j \mathbf{r}'_j = 0$, equation (2.6) is reduced to:

$$\mathbf{L} = M\mathbf{R}_{CM} \times \mathbf{v}_{CM} + \sum_j m_j \mathbf{r}'_j \times \mathbf{v}'_j . \tag{2.7}$$

Therefore, the angular momentum consists of a sum of the angular momentum of the center of mass (first term in r.h.s. of 2.7) and the angular momentum about the center of mass (second term in r.h.s. of 2.7). If the center of mass is chosen to be the origin of coordinates ($\mathbf{R}_{cm}$=0), (which can be done without loss of generality), then:

$$\mathbf{L}\big|_{R_{CM}} = \sum_j m_j \mathbf{r}'_j \times \mathbf{v}'_j . \tag{2.8}$$

**3.**

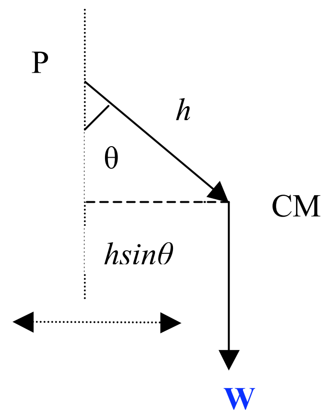


***Fig 1. Problem 3***

The dynamics of the disk follows the second law for rigid bodies: $\sum_j \vec{\tau}_j = I\vec{\alpha}$, where I is the moment of inertia with respect to a rotation axis passing through P, α is the angular acceleration of the object ($\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$) and τ corresponds to the torque induced by external forces. The only force that generates torque in this case is the weight of the disk W and the equation for the angular amplitude θ is given by

$$-hW\sin\theta = \left(Icm + Mh^2\right)\ddot{\theta}, \tag{3.1}$$

where the theorem of parallel axis has been used. In the small-angle approximation, sinθ≈θ and (3.1) becomes into the SHO equation in the variable θ:

$$\ddot{\theta} = -\frac{hW}{\left(Icm + Mh^2\right)}\theta = -\omega^2\theta. \tag{3.2}$$

The frequency of vibration is therefore:

$$\omega = \sqrt{\frac{hW}{\left(Icm + Mh^2\right)}} = \sqrt{\frac{hMg}{\left(\frac{MR^2}{2} + Mh^2\right)}}. \tag{3.3}$$

The period of is obtained from (3.3):

$$T = 2\pi\sqrt{\frac{R^2/h + 2h}{2g}}, \tag{3.4}$$

This period is a minimum if $\frac{dT}{dh}=0$ and $\frac{d^2T}{dh^2}>0$. The first condition gives $2\pi(\frac{1}{2})(\frac{1}{g})(\frac{-R^2}{h^2}+2)=0$ so that $h=\pm R/\sqrt{2}$, but the negative value can be discarded since it corresponds to a suspension point below the centre of mass and an imaginary period). For this value of h one has $\frac{d^2T}{dh^2}=2\pi(\frac{1}{2})(\frac{1}{g})(\frac{2R^2}{h^3})>0$ and so the value of $h=R/\sqrt{2}$ gives a minimum period. The minimum value of the period is therefore: $T_{min}=2\pi\sqrt{\sqrt{2}R/g}$.

**4**. We can solve this problem by considering the net forces and the net torques which act on the sphere. If $F_f$ is the frictional force acting on the bottom of the sphere, then we have that Newton's second law for the sphere net force acting on the sphere following a displacement $x$ is

$$M\frac{d^2x}{dt^2}=-2kx+F_f \tag{4.1}$$

The net torque about the centre of mass of the sphere arises from friction only since the spring induces no torque about the sphere centre. We then have

$$F_fR=I_{cm}\alpha \tag{4.2}$$

where $\alpha$ is the angular acceleration. From the no slip condition of a point on the bottom of the sphere we have

$$\alpha R+a_{cm}=\alpha R+\frac{d^2x}{dt^2}=0 \tag{4.3}$$

Combining 4.1, 4.2 and 4.3 gives (with $I_{cm}=\frac{2}{5}MR^2$)

$$M(1+\frac{2}{5})\frac{d^2x}{dt^2}=-2kx \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{10k}{7M}x$$

so that $\omega_0=\sqrt{\frac{10k}{7M}}$.

One could also use the principle of conservation for the total (mechanical) energy:

$$E=K_{rot}+K_{trans}+U_{elast}, \tag{4.4}$$

where $K_{rot}$ is the rotational kinetic energy, $K_{trans}$ corresponds to the translational kinetic energy associated to the center of mass, and $U_{elast}$ is potential energy of the springs. Explicitly:

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}kx^2 + \frac{1}{2}kx^2 \tag{4.5}$$

The speed of the center of mass of the sphere is related with its angular speed ω through the *non-slip* condition: $v_{CM} = R\omega = \dot{x}$. Equation (4.5) is transformed into:

$$E = \frac{1}{2}\left(\frac{I_{cm}}{R^2} + M\right)\left(\dot{x}\right)^2 + \frac{1}{2}(2k)x^2. \tag{4.6}$$

Taking the time-derivative of the total energy, we get:

$$\frac{dE}{dt} = \left(\frac{I_{cm}}{R^2} + M\right)\left(\dot{x}\,\ddot{x}\right) + (2k)x\,\dot{x} = 0, \tag{4.7}$$

since E=constant. Equation (4.7) has two solutions: (i) $\dot{x} = 0$, which indicates that the system remains in rest, and (ii)

$$\left(\frac{I_{cm}}{R^2} + M\right)\ddot{x} + 2kx = 0 \tag{4.8}$$

which is of the form $\ddot{x} = -\omega_0^2 x$, with the frequency given by:

$$\omega_0 = \sqrt{\frac{10K}{7M}}. \tag{4.9}$$

The period T for k = 500 N/m, M = 10 kg and R = 0.1 m, is:

$$T = 2\pi\sqrt{\frac{\left(\frac{I}{R^2} + M\right)}{2K}} = 2\pi\sqrt{\frac{7M}{10k}} = 2\pi\sqrt{\frac{7 \times 10}{10 \times 500}} = 0.74s$$