Solution to Problem Set 7

1. (a) Let $x(t)$ be the length of the vertical chain at any time; the mass of the chain is $m = \mu x(t)$, and the center of mass is located at $x/2$. The speed of the center of mass is therefore $v_{cm} = 1/2\ v$ where $v = dx/dt$ is the speed of the chain. The net force acting on the chain is $F_{net} = F_{app} - m(t)g$, where F_{net} is the net force, and F_{app} is the applied force.

Hence, by Newton's second law for a system $F_{net} = F_{app} - \mu x(t)\, g = dP_{cm}/dt$ where $P_{cm} = mv_{cm} = \mu x(t)\, v/2$.

Therefore, $F_{app} - \mu x(t)\, g = m dv_{cm}/dt + v_{cm} dm/dt = 0 + v_{cm}(\mu v) = 1/2 \mu v^2$

If we take $x(0)=0$ then $x(t) = vt$ and $\boldsymbol{F_{app} = 1/2\mu v^2 + \mu vgt}$**.**

(b) Use conservation of energy. (You can also use force concepts, but the solution is more difficult. Besides you're not asked for any time dependence.) Taking the table as the zero for potential energy, the initial potential energy is that of the chain hanging over the table.

$U_i = mgh = (L\mu/4)\, g\, (-L/8)$

where $h = -L/8$ is the displacement of the center of the mass of the part of the chain hanging from the table. When the entire chain is off the table the final gravitational potential energy

$U_f = (L\mu) g (-L/2)$

Let v be the speed of the (entire) chain at this point. Conservation of energy then implies

$\mu L v^2/2 = U_i - U_f = 15\mu L^2 g/32$ so that $\mathbf{v} = \dfrac{\sqrt{15 gL}}{4}$.

2. (a) The average force $\underline{F} = \Delta P/\Delta t$. During a duration of $\Delta t = 60$ s, for the bullets ΔP = 240 bullets* 0.025 (kg/bullet) * 900 (m/s). = 6 * 900 Ns. Hence, $\boldsymbol{\underline{F} = 90\ N}$.

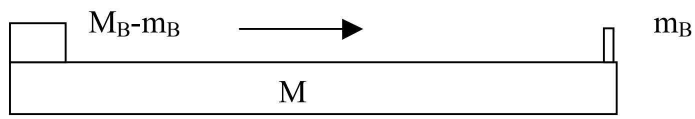
(b) There are several ways to make the estimate. As long as one can state one's assumptions, a case can be made. I will consider what happens to a single bullet. The momentum gain ΔP_1 for one bullet is:

ΔP_1 = (0.025 kg)*900 m/s = 23 (Ns) and the average force experienced by the single bullet in the gun is

$\underline{F} = \Delta P/\Delta t$ = (22.5 Ns) / (2 * 10^{-3}) = **1.1 * 10^4 N.**

The real force is a complicated function of time. It likely peaks early and decays over 2 ms while the bullet is in the gun. If we assume a triangular shape for the force as a function of time the peak force is approximately twice the average force on a single bullet or about **2 x10^4 N**. By Newton's third law this is the same as the peak force on the gun.

(c) Because the entire system is frictionless, there is not net external force on it and the center of mass of the system remains stationary. However, as bullets move (are fired) from one end towards the other end, mass is transferred and the rest of the system (flatcar plus machine gun, etc.) must move to keep the center of mass fixed.



At any time let the total mass of all the bullets be M_B and the mass of bullets at the left of the flatcar be m_B. Let the rest of the system have mass M so that the total mass of the system $M+M_B = 10^4$ kg. If the length of the car is L the center of mass of the system is given by

$$x_{cm} = \frac{1}{M+M_B}\left[Mx_{car} + (M_B - m_B)(x_{car} - L/2) + m_B(x_{car} + L/2)\right]$$

where x_{car} is the center of mass of the flatcar. Taking a time derivative, we have

$$\frac{dx_{cm}}{dt} = 0 = v_{car} + \frac{L}{M+M_B}\frac{d}{dt}m_B \text{ or } v_{car} = -\frac{L}{M+m}\frac{dm_B}{dt} =$$

$$-\frac{10m}{10^4\,kg}(240/min)(.025kg) = \mathbf{-10^{-4}\ m/s}$$

The minus sign means the car moves in the direction of the gun.

3. (a) The implied condition is that α, β > 0, r > 0, and n is a positive integer.

$U(r) = -\alpha r^{-n} + \beta r^{-2}$.

Hence, $dU/dr = +\alpha n r^{-n-1} - 2\beta r^{-3}$.

For equilibrium we require $F = -dU/dr = 0$ or $\alpha n r_0^{-n+2} = 2\beta r_0$

Giving $r_0 = \left(\frac{\alpha n}{2\beta}\right)^{1/(n-2)}$, $n \neq 2$.

For n = 2, $U = (\beta - \alpha) r^{-2}$, and U has no extrema.

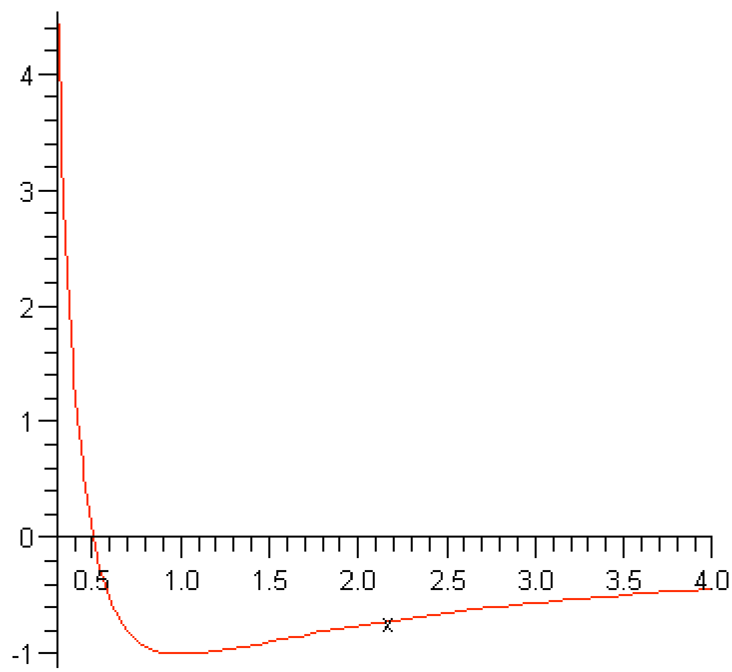
(b) An equilibrium position can be stable, neutral or unstable if the second derivative of the potential is positive, zero or negative. For stable equilibrium we require

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} > 0 \text{ . We have } \left.\frac{d^2U}{dr^2}\right|_{r=r_0} = -\frac{n(n+1)\alpha}{r_0^2} + \frac{6\beta}{r_0^4} > 0 \text{ or } -\frac{n(n+1)\alpha}{r_0^{n-2}} + 6\beta > 0$$

Substitution of the equilibrium position gives $-\frac{n(n+1)\alpha}{n\alpha/2\beta} + 6\beta > 0$. Hence $-2(n+1)+6 > 0$, $-(n+1) > -3$, $n+1 < 3$ or $n < 2$.

(c) For $\alpha = 2\beta$, $n = 1$, $U(r) = -2\beta r^{-1} + \beta r^{-2}$.

U(r) goes to $+\infty$ as r goes to 0^+. U(r) goes to 0 as r goes to $+\infty$. U(r) has a local minimum at r = 1, and has a zero at r = ½. Note that these values are independent of β. We then have the following plot:



The plot is computer-generated with β =1, but changing the value of β only affects the exact value of the extrema, but not the locations of them.

4. (a) If we assume the exhaust velocity is constant as we always have the momentum of the exhaust is given as(taking upwards as positive), we have from the rocket equation, including the gravitation force,

$$M\frac{dv}{dt} = -v_e\frac{dM}{dt} - Mg$$

For lift off we need $\frac{dv}{dt} > 0$ or $-v_e\frac{dM}{dt} - Mg > 0$

In this particular problem, M = 10^4 kg. dM/dt = - 20kg/s. Therefore $v_e \geq 9.8 * 10^4 / (20)$ = **$4.9 * 10^3$ m/s.**

(b) Neglecting gravity, the rocket equation $Mdv = -v_e dM$ integrates to

$v_f = v_i + v_e \ln\frac{M_i}{M_f}$. Taking the initial speed to be zero we have

v_f*=4900m/s ln10=**11 km/s** i*f there is 9000 kg of fuel (1000kg of payload).

If the payload is 100 kg v_f*=4900m/s ln100=**23 km/s**.*