Solution for Problem Set #5

1.

<u>To derive expressions for A and ϕ</u>

To get A and ϕ, we can start with the original differential equation that describes this motion:

$m(d^2x/dt^2) = -b(dx/dt) - kx + F_o\cos(\omega t)$ (1)

$\therefore (d^2x/dt^2) + (b/m)(dx/dt) + (k/m)x - (F_o/m)\cos(\omega t) = 0$ (2)

As the solution for Eq. 2 is $x = A\cos(\omega t + \phi)$, (3)

$\therefore dx/dt = -A\omega\sin(\omega t + \phi)$ (4)

& $d^2x/dt^2 = -A\omega^2\cos(\omega t + \phi)$. (5)

Set $c_1 = b/m$, $c_2 = k/m$, and $c_3 = F_o/m$. (6a -6c)

Taking (6) to (2):

$(d^2x/dt^2) + c_1(dx/dt) + c_2 x - c_3\cos(\omega t) = 0$ (7)

Taking (3)–(5) to (7):

$-A\omega^2\cos(\omega t + \phi) - c_1A\omega\sin(\omega t + \phi) + c_2A\cos(\omega t + \phi) - c_3\cos(\omega t) = 0$ (8)

Or $A\omega^2\cos(\omega t + \phi) + c_1A\omega\sin(\omega t + \phi) - c_2A\cos(\omega t + \phi) + c_3\cos(\omega t) = 0$ (9)

Using :

$\mathrm{Sin}(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\mathrm{Cos}((\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ in (9):

$(A\omega^2 - c_2A)(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi) + c_1A\omega(\sin(\omega t)\cos\phi + \cos\omega t\sin\phi) + c_3\cos(\omega t) = 0.$ (10)

Combining the coefficients for the $\cos(\omega t)$ and $\sin(\omega t)$ separately:

$[(A\omega^2 - c_2A)\cos\phi + c_1A\omega\sin\phi + c_3]\cos(\omega t) + [-(A\omega^2 - c_2A)\sin\phi + c_1A\omega\cos\phi]\sin(\omega t) = 0$ (11)

$\therefore$ Equ. (11) is satisfied if the coefficients are zero.

$(A\omega^2 - c_2A)\cos\phi + c_1A\omega\sin\phi + c_3 = 0$ (12)

$-(A\omega^2 - c_2A)\sin\phi + c_1A\omega\cos\phi = 0$ (13)

We can solve (12) and (13) to get A and ϕ.

From (13):

$\tan\phi = c_1\omega/(\omega^2-c_2)$ (14)

From (6): $c_1=b/m$, $c_2=k/m=\omega_o^2$:

$\therefore$ ***$\tan\phi = (b\omega/m)/(\omega^2-\omega_o^2)$*** (15)

Taking (15) to (12):

$A[(\omega^2-c_2)\cos\phi + c_1\omega\sin\phi]+ c_3 =0$

$\therefore A= - c_3 /[(\omega^2-c_2)\cos\phi + c_1\omega\sin\phi]$ (16)

Taking:

$\cos\alpha= \pm 1/(1+ \tan^2\alpha)^{1/2}$ (17)

$\sin\alpha=\tan\alpha\cos\alpha$ (18)

to (16):

$A= - c_3 /[(\omega^2-c_2)(1+ \tan^2\alpha)^{-1/2} + c_1\omega \tan\alpha(1+ \tan^2\alpha)^{-1/2}]$

or

$A= - (F_o/m)(1+ \tan^2\alpha)^{1/2} /[(\omega^2-\omega_o^2)+ (b/m)\omega \tan\alpha]$ (19)

$\therefore$ The numerator in (19) equals

$- (F_o/m)[1+ \tan^2\alpha]^{1/2} = - (F_o/m)[(\omega^2-\omega_o^2)^2 +(b\omega/m)^2]^{1/2} /(\omega^2-\omega_o^2)$ (20)

$\therefore$ The denominator in (19) equals

$[(\omega^2-\omega_o^2)+ (b/m)\omega \tan\alpha]= [(\omega^2-\omega_o^2)^2+ (b\omega/m)^2]/(\omega^2-\omega_o^2)$ (21)

Taking (20) and (21) to (19) (taking an absolute value: positive sign):

$\therefore$ ***$A = (F_o/m)/[(\omega^2 - \omega_o^2)^2 + (b\omega/m)^2]^{1/2}$*** (22)

<u>To derive expressions for P:</u>

$P(t)=F(t)v(t)$

The average power delivered to the system by the driving force over one cycle is:

$$p = \frac{1}{\tau}\int_0^\tau \mathrm{F(t)v(t)dt} = \frac{1}{\tau}\int_0^\tau -\mathrm{F_0}\cos(\omega t)\mathrm{A}\omega\sin(\omega t+\phi)\mathrm{dt}$$

where $\tau=2\pi/\omega$

$$P = \frac{F_0 A\omega}{\tau}\int_0^\tau -\{\cos(\omega t)\sin(\omega t)\cos\phi + \cos(\omega t)\cos(\omega t)\sin\phi\}dt$$

$$P = \frac{A\omega F_0}{2\pi}\{\int_0^{2\pi/\omega} \cos\phi \cos(\omega t) d\cos(\omega t) - \int_0^{2\pi/\omega} \sin\phi \cos^2(\omega t) dt\}$$

$$P = \frac{A\omega F_0}{2\pi}\{\cos\phi[\cos^2(2\pi) - \cos^2(0)] - [\sin\phi(\frac{1}{2} * 2\pi) + \sin\phi * \frac{1}{4}\sin(4\pi)]\}$$

$$\therefore P = \frac{-A\omega F_0}{2}\sin\phi \qquad (23)$$

Let $\theta=\phi-\pi/2$

$$P = \frac{A\omega F_0}{2}\cos\theta$$

From (22): $A = (F_o/m)/[(\omega^2 - \omega_o^2)^2 + (b\omega/m)^2]^{1/2}$

From (15): $\tan\phi = (b\omega/m)/(\omega^2-\omega_o^2)$

$$\mathrm{Sin}\phi = \frac{1}{\sqrt{ctg^2\phi + 1}} = \frac{1}{\sqrt{(b\omega/m)^{-2}(\omega^2 - \omega_0^{\ 2}) + 1}} = \frac{b\omega/m}{\sqrt{(\omega^2 - \omega_0^{\ 2}) + (b\omega/m)^2}}$$

From (23) (taking a positive sign, as the negative sign is cancelled in Equ. (19)):

$$\therefore \mathbf{P = (b/2)(A\omega)^2=(b/2)v_{max}^2} \qquad \mathbf{(24)}$$

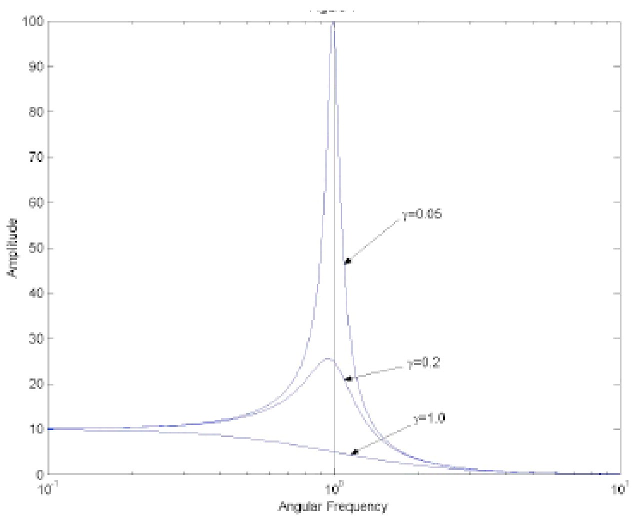
In conclusion:

1) The expression for the amplitude is the same as Eq. (15.36)
$A = (F_o/m)/[(\omega^2 - \omega_o^2)^2 + (b\omega/m)^2]^{1/2}$
where $\omega_o^2 = k/m$

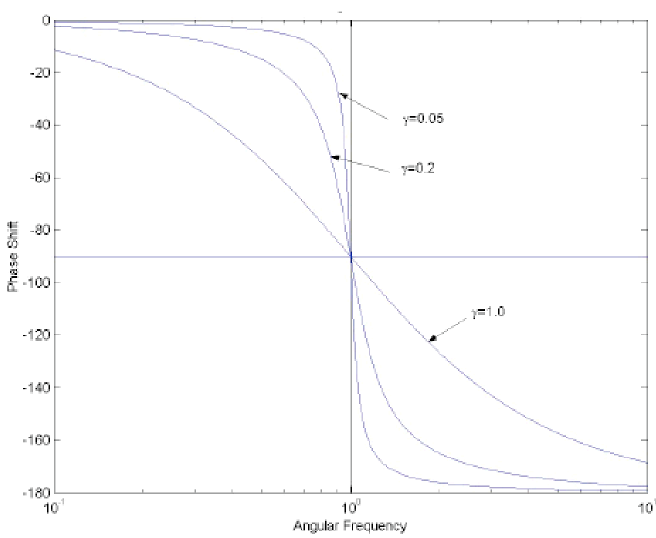
2) The phase constant is dependent on ω in the following way:
$\tan\phi=(b\omega/m)/ (\omega^2 - \omega_o^2)$

3) The average power delivered to the system over one cycle is:
$P = (b/2)(A\omega)^2=(b/2)v_{max}^2$

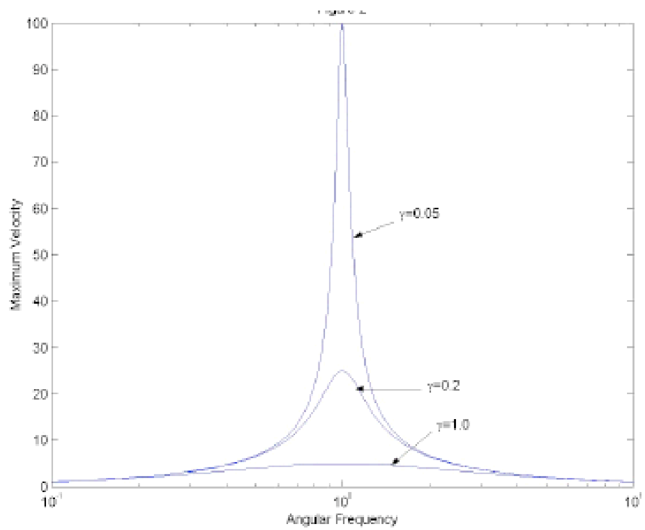
The graphs for A, ϕ, and P are shown below. In the graphs, γ is defined as $\gamma = b/(2m)$ and $\omega_o = 1$.The y-axis is relative value for A and P (in term of v_{max}) and degree for ϕ.



Amplitude
Angular Frequency
γ=0.05
γ=0.2
γ=1.0

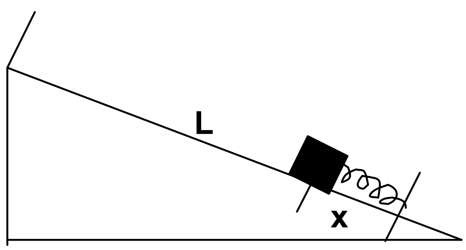


Phase Shift
Angular Frequency
γ=0.05
γ=0.2
γ=1.0



2.

m=3 kg
α=30°
k=90 N/m
L=5 m
g=9.8 m/s^2
μ=0.2



a) How much energy is stored in the spring at maximum compression?

Let us divide the total distance in two, one is L which is from the original position of the block to the location when the block hits the spring, and the other is x that is from the equilibrium location of the end of the spring to its maximally compressed location. According to the law of energy conservation:

$(E_1-W_1) +(E_2-W_2)=E_{spring}$
$E_1= mg(L\sin\alpha)=3*9.8*5*\sin30°=73.5$ J
$W_1= \mu(mg\cos\alpha)L=0.2*3*9.8*\cos30°*5 =25.5$ J
$E_2= mg(x\sin\alpha)=3*9.8*\sin30°\ x=14.7x$
$W_2= \mu(mg\cos\alpha)x =0.2*3*9.8*\cos30°\ x =5.09x$
$E_{spring}=k\ x^2/2$
$\therefore (90/2)\ x^2-(14.7-5.09)x -(73.5-25.5)=0$
$\therefore 45x^2-9.6x - 48 =0$
$\therefore x =1.15$ (m) (discarding x= -0.93 m)

$E_{spring}= k\,x^2/2 = 90 * 1.15^2/2 = 59.51$ J
$\therefore$59.5 J is stored in the spring.

b) What is the maximum distance back up the incline that the block travels
Let D be the maximum distance, according to the law of energy conservation:
$k\,x^2/2 - W = E$
$k\,x^2/2 = E+W=mg(D\sin\alpha) + (\mu mg\cos\alpha)\,D$
$D = kx^2/(2*mg\,(\sin\alpha + \mu\cos\alpha)) = 90 * 1.15^2/(2*3*9.8*(\sin 30° + 0.2* \cos 30°))$
$D = 3.1$ (m)
$\therefore$The maximum distance is 3.1 m.

c) What is the power delivered to the block when the spring is compressed by half its maximum amount while the block is traveling back up the incline?
Taking upward along the incline as positive:
$F= kx/2 =51.8$ (N)
According to the law of energy conservation:
$mv^2/2 + mg\,(x/2)\sin\alpha+ k(x/2)^2/2 = kx^2/2 - (\mu mg\cos\alpha)*(x/2)$
$\therefore\ v^2 = (2/m)* (kx^2/2- kx^2/8- (\mu mg\cos\alpha)*(x/2) - mg\,(x/2)\sin\alpha\,)$
$\therefore\ v = 4.69$ (m/s)
$\therefore\ P = Fv = 243$ (w)
$\therefore$ The power delivered to the block is 243 W.

3.
$m_p=3.1\ g=3.1*10^{-3}$ kg
$m_b=20.0\ g=20.0*10^{-3}$ kg
$\mu_{s\_b}=0.75$
$\mu_{k\_b}=0.64$
$\mu_{s\_p}=0.52$
$\mu_{k\_p}=0.45$
$r =12.0$ cm$=0.12$ m
$g=9.8$ m/s^2

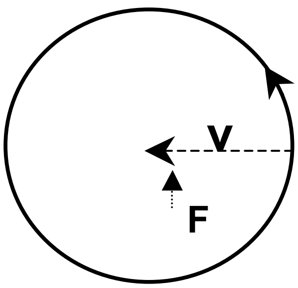
Centripetal force: $F=ma$
$F=\mu N= \mu mg$
Let maximum frequency to keep an object from sliding be f: $a=r\omega^2=4\pi^2rf^2$
$\therefore\mu g=4\pi^2rf^2$
$\therefore\ f^2=\mu g/ (4\pi^2r)$
Only static friction coefficients are useful in this question since nothing moves.
To keep the block from sliding, use $\mu_{s\_b}=0.75$
$f_b^2=\mu g/ (4\pi^2r)=0.75*9.8/(4\pi^2*0.12)=1.55$ (s^{-2})
$\therefore\ f_b=1.25$ (s^{-1})

To keep the penny from sliding on the block, use $\mu_{s\_p}=0.52$
$f_p^2=\mu g/ (4\pi^2r)=0.52*9.8/(4\pi^2*0.12)=1.08$ (s^{-2})

$\therefore\ f_p = 1.04\ (s^{-1})$

$f = \min(f_b, f_p)$
$\therefore\ f = 1.04$ (revolutions/s)
$f = 62.2$ (revolutions/min)
$\therefore$ The maximum rate of rotation is 62.2 rev. per minute.

4.
$R = 5$ m
$f = 1$ rev/5 s = 0.2 rev/s
$v = 2$ m/s



To Samantha who is in a rotating frame, the ball will received a fictitious force -- the Coriolis force (F_c), which is perpendicular to the velocity. Therefore, the ball will have an acceleration perpendicular to the velocity.
$F_c = 2\omega vm$
$\therefore 2\omega vm = ma$
$\therefore\ a = 2\omega v = 4\pi fv$
$t = R/v = 5/2 = 2.5$ (s)
$\therefore$ The distance is:
$d = (1/2)at^2 = \omega v\ (R/v)^2 = = \omega R^2/v = (2\pi*0.2)(5)^2/2 = 15$ m
Direction is as shown on the above figure. If the merry-go-round is turning in an opposite direction, the Coriolis force will be in an opposite direction. Therefore, the ball will miss Samantha by 15 m due to the rotation of the merry-go-round.

One can also solve this in the "ground" frame. According to this frame the ball has a tangential (sideways) speed of $R\omega$ and it travels for a time $=R/v$ before reaching a position opposite the center. Therefore the "sideways" distance it travels is $R\omega\ (R/v) = \omega R^2/v$ as before.