Solutions
Problem set # 4
(Prepared by H. van Driel)

1. The train experiences an acceleration at any time given by $\vec{a} = \frac{dv}{dt}\hat{\theta} - \frac{v^2}{R}\hat{r}$ where v is the speed of the train (= tangential speed). We require that the magnitude of the acceleration $a = \sqrt{(dv/dt) + (v(t)^2/R)^2} \le 0.2g$. For constant tangential acceleration with $a_t = \frac{dv}{dt} = \frac{v_f - v_i}{t}$ where *t* is time over which the train accelerates, we have that at any time that $\sqrt{a_t^2 + (v(t)^2/R)^2} \le 0.2g$. Within the square root only the centripetal acceleration, $v(t)^2/R$, is changing and its maximum value is v_f^2/R. Therefore we must have $a_t^2 < .04g^2 - v_f^2/R = 2.4$ m^2/s^4 or $a_t < 1.5$ m/s^2. From $a_t = \frac{v_f - v_i}{t}$, t > 10 s for v_f = *300* km/hr =*83 m/s* with v_i =*250* km/hr =*69* m/s. The minimum time is ~ 10 s.

2. i) For the first situation, we have that if the displacement of the block from equilibrium is taken as *x*, then the spring on the left is extended (if *x* is positive) and exerts a restoring force towards the left, while the spring on the right is compressed and exerts a force to the left as well. The total force on the mass is $F_1 + F_2 = -k_1 x - k_2 x = -(k_1 + k_2)x = M\frac{d^2x}{dt^2}$ and so the angular or natural/resonance frequency of the system is $\omega_0 = \sqrt{\frac{k_1 + k_2}{M}}$. In the second situation, the force equation is the same, and so the two systems are dynamically equivalent and have the same resonance frequency.

ii) If the spring originally with length L is extended by x by a force *F* its new length becomes $L + x = L + F/k$. A point in the middle of the spring therefore moves from *L/2* to $(L + x)/2$ so it has moved $x/2$. When the spring is extended, by Newton's third law any point (including the midpoint) in the massless spring experiences the same tension or restoring force, F, to the right and to the left. Hence if half the spring has constant k_{eff} we have $x/2 = F/k_{eff}$ so that $k_{eff} = 2k$.

For the H_2 molecule, both atoms oscillate at the same frequency (by symmetry) and are accelerating at all times (except when they pass through their equilibrium position). Therefore neither serves as an inertial frame of reference to apply Newton's second law. The only point that can serve as a reference point for an inertial frame is (by symmetry) the midpoint of the bond (for a more complex molecule the point would be the center of mass). Therefore each atom is vibrating against the center point with half the "bond spring". If the "bond spring" has constant *k,* half the "bond spring" has constant *2k* (see above) and we have the resonance or natural frequency of each atom is $\omega = \sqrt{2k/m_{proton}}$. For m_{proton}= 1.7x10^{-27}kg, we have k = 260 N/M.

3. The equilibrium position is given by F=0. This gives $x_{eq} = C/B$= 5 cm. If we define the displacement relative to the equilibrium position to be $X = x - x_{eq}$, then we can write the equation of motion as $R\frac{d^2X}{dt^2} = -BX$ where we used the fact that $\frac{d^2x_{eq}}{dt^2} = 0$ (derivative of a constant is zero). This equation has the general solution $X = A\cos(\omega_0 t + \phi)$ where $\omega_0 = \sqrt{B/R} = \sqrt{10}$ /s. Then $x = x_{eq} + A\cos(\omega_0 t + \phi)$ and $v_x = \frac{dx}{dt} = -\omega_0 A\sin(\omega_0 t + \phi)$. Applying x = 0.5 m at t = 0, and v_x= -1 m/s at t= 1 s, we have two equations in two unknowns (A, ϕ), namely, 0.5 =.05 +Acosϕ, and -1 = $-\sqrt{10}A\sin(\sqrt{10} + \phi)$. Eliminating A we have an equation for ϕ. Paying attention to where we are in the cycle we find ϕ = –0.6 and A= 0.6 m. We then have that the acceleration is $\frac{d^2x}{dt^2} = \frac{d^2X}{dt^2} = -\frac{2}{0.2}(0.6)\cos\left(3\sqrt{10} - 0.63\right)$ = 5. m/s^2.

4. For the block moving up the plane the equation of motion along the plane is:

$$(1)\ M\frac{d^2L}{dt^2} = -Mg\sin\theta - Mg\cos\theta(0.1 + 0.03L) = B - DL \text{ where}$$
$$B = -Mg\sin\theta - 0.1Mg\cos\theta$$
$$D = 0.03Mg\cos\theta.$$

We saw from problem 3 that the general solution to equation (1) is $L = B/D + A\cos(\omega_0 t + \phi)$ or $L = -37\ m + A\cos(0.46t + \phi)$ (using g=10 m/s^2, θ =π/4, $\omega_0 = \sqrt{D/M}$). Using the condition that the object starts at L = 0 at t = 0 with v = $-A\omega_0\sin(\omega_0 t + \phi)$= 5 m/s at t = 0, we can find A and ϕ; A= 38.5 m and ϕ = –0.28. The block reaches its maximum position up the plane when the cosine function's argument is zero, so that L_{max}=1.5 m.

For the block moving down the plane the equation of motion is

$$M\frac{d^2L}{dt^2} = -Mg\sin\theta + Mg\cos\theta(0.1 + 0.03L) = B' + DL \text{ where}$$
$$B' = -Mg\sin\theta + 0.1Mg\cos\theta$$

Taking a hint from what we did in Q3 and the earlier part of this problem, let's define *L'=L-B'/D.* Then the equation of motion becomes $\frac{d^2L'}{dt^2} = \frac{D}{M}L'$. But the only functions whose second derivative returns the function is the exponential function, $e^{\pm t}$. The general solution to this equation is $L' = Ce^{t\sqrt{D/M}} + Ge^{-t\sqrt{D/M}}$ where C, G are constants. We then have that $L = \frac{B'}{D} + Ce^{0.46t} + Ge^{-0.46t} = -30m + Ce^{0.46t} + Ge^{-0.46t}$. But since $L = L_{max}$ and v = 0 at t = 0, we calculate that C=G= 15.75 m. When L= 0 (block back at bottom), t ~ 0.66 s and v = -4.6 m/s. The speed of the block is therefore 4.6 m/s at the bottom.

Yes, you can also do this using conservation of energy, something we haven't covered yet. That technique also requires that you known integration.