Problem # 1

i)

For a circle we require that $r = \text{constant}$, which means that

$$\frac{1}{1-e\cos\theta} = const. \Rightarrow e = 0$$

For an ellipse we require that $0 < r < \propto$, which means that

$$0 < \frac{1}{1-e\cos\theta} < \propto, \Rightarrow 1-e\cos\theta > 0; 1 > e\left|\cos\theta\right|_{\max} \to e < 1$$

ii)

$$r = \sqrt{x^2+y^2}, \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore r = \frac{L}{(1-e\cos\theta)} \Rightarrow \sqrt{x^2+y^2} = \frac{L}{\left(1-e\frac{x}{\sqrt{x^2+y^2}}\right)}$$

$$\sqrt{x^2+y^2} - ex = L \Rightarrow \sqrt{x^2+y^2} = L + ex$$

$$\therefore x^2 + y^2 = L^2 + 2eLx + e^2x^2$$

$$(1-e^2)x^2 - 2eLx + y^2 = L^2$$

$$x^2 - 2\frac{eL}{(1-e^2)}x + \frac{1}{(1-e^2)}y^2 = \frac{L^2}{(1-e^2)}$$

$$\left[x - \frac{eL}{(1-e^2)}\right]^2 + \frac{1}{(1-e^2)}y^2 = \frac{L^2}{(1-e^2)^2}$$

$$X^2 + \frac{1}{(1-e^2)}Y^2 = \frac{L^2}{(1-e^2)^2}$$

$$\frac{(1-e^2)^2}{L^2}X^2 + \frac{(1-e^2)}{L^2}Y^2 = 1 \quad (*)$$

where

$$\frac{(1-e^2)^2}{L^2}=\frac{1}{a^2}\Rightarrow a=\frac{L}{(1-e^2)} \text{ - semi-major axis}$$

$$\frac{(1-e^2)}{L^2}=\frac{1}{b^2}\Rightarrow b=\frac{L}{\sqrt{(1-e^2)}} \text{ - semi-minor axis}$$

$$L=\frac{b^2}{a};e=\sqrt{1-\frac{b^2}{a^2}}=\frac{\sqrt{a^2-b^2}}{a}=\frac{c}{a}$$

iii)

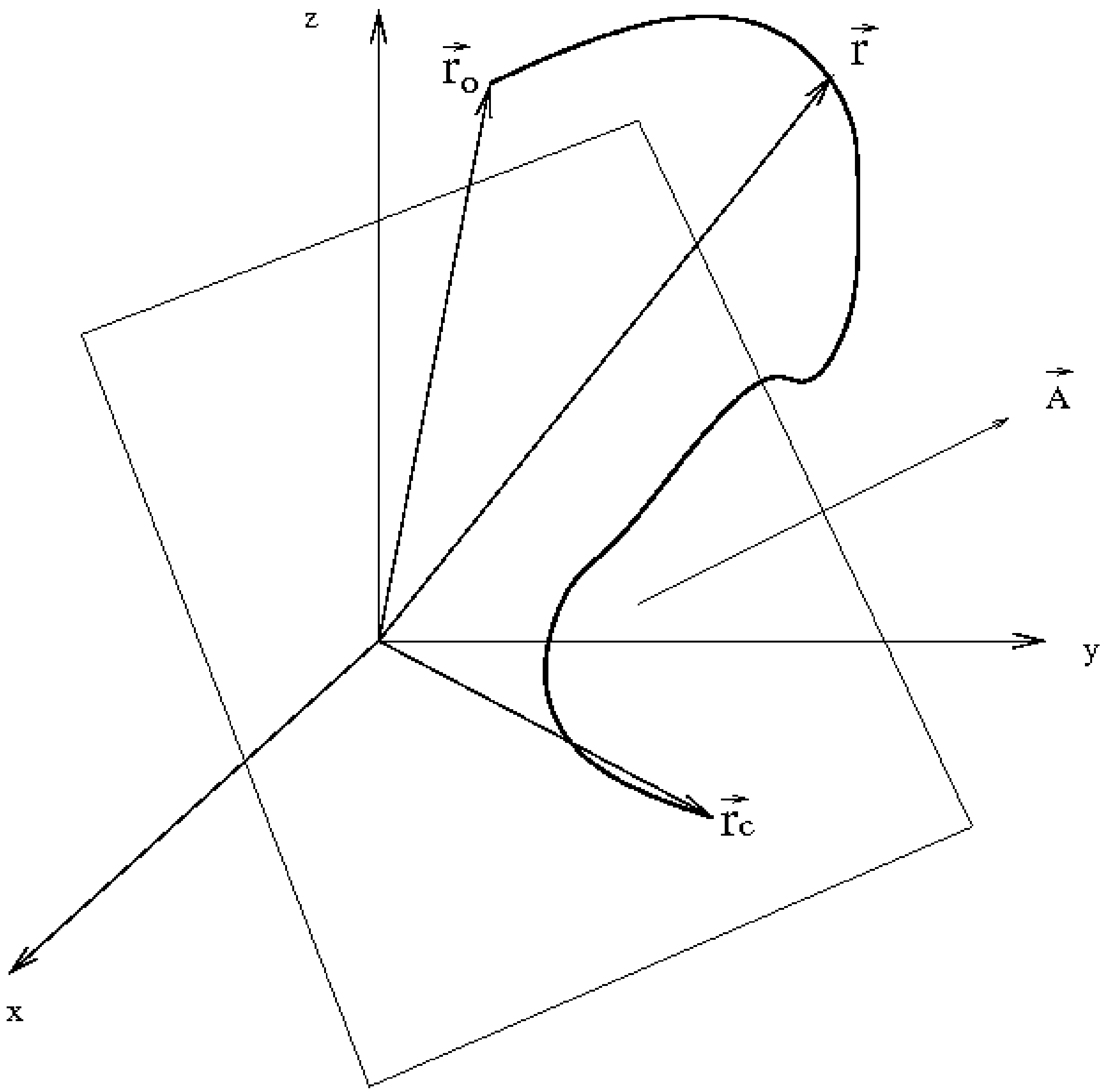
Let AP="apogee" and PR="perigee". Then (see Fig. 13.5 from your textbook)

$$a=\frac{1}{2}(AP+PR);c=a-PR$$

$$\therefore e_{\text{Pluto}}=\frac{c}{a}=\frac{a-PR}{a}=1-\frac{PR}{a}=\frac{AP-PR}{AP+PR}$$

$$e_{\text{Pluto}}=\frac{3.0}{11.8}=0.25 \quad (**)$$

Problem # 2



$$\vec{r} = (1+t^2)\hat{i} + (-4t+3)\,\hat{j} + (1-t^2)\hat{k}$$

$$\vec{r} = x_F\hat{i} + y_F\,\hat{j} + z_F\hat{k}$$

$$\upsilon_F = \frac{d\vec{r}}{dt} = (2t)\hat{i} + (-4)\,\hat{j} + (-2t)\hat{k}$$

a)

Fly lands when $x_F + y_F + 10z_F = 5$ which leads to the following quadratic equation

$$9t^2 + 4t - 9 = 0$$

$$t_{1,2} = \frac{\left(-2 \pm \sqrt{85}\right)}{9} = \{-1.25, 0.80\}$$

And we have to take the positive root, since the fly begins at $t = 0$. Therefore the fly lands at 0.80 time units.

b)

Velocity is perpendicular to the table when

$$\vec{A} = (1,1,10) \Rightarrow \vec{A} \times \vec{v}_F = \vec{0}$$

$$\vec{A} \times \vec{v}_F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 10 \\ 2t & -4 & -2t \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\hat{i}: \quad -2t + 40 = 0 \Rightarrow t = 20$$

$$\hat{j}: \quad 22t = 0 \Rightarrow t = 0$$

$$\hat{k}: \quad -20t - 4 = 0 \Rightarrow t = -\frac{1}{5}$$

And since the solutions from the three linear equations do not coincide, we can conclude that the velocity of the fly is never perpendicular to the table.

c)

The shadow of the fly is

$$\vec{r}_c = x_F\hat{i} + y_F\hat{j} + z_c\hat{k} = (1 + t^2)\hat{i} + (-4t + 3)\hat{j} + z_c\hat{k}$$

And also

$$x_F + y_F + z_c = 5 \Rightarrow (1 + t^2) + (-4t + 3) + 10z_c = 5$$

Now solve for z_c and replace in $\vec{r}_c$

$$z_c = \frac{1}{10}(5 - x_F - y_F) = \frac{1}{10}\left(-t^2 + 4t + 1\right)$$

$$\therefore \vec{r}_c = (1 + t^2)\hat{i} + (-4t + 3)\hat{j} + \frac{1}{10}\left(-t^2 + 4t + 1\right)\hat{k}$$

$$\vec{v}_c = (2t)\hat{i} + (-4)\hat{j} + \frac{1}{5}(-t + 2)\hat{k}$$

So that the speed of the fly's shadow along the table is:

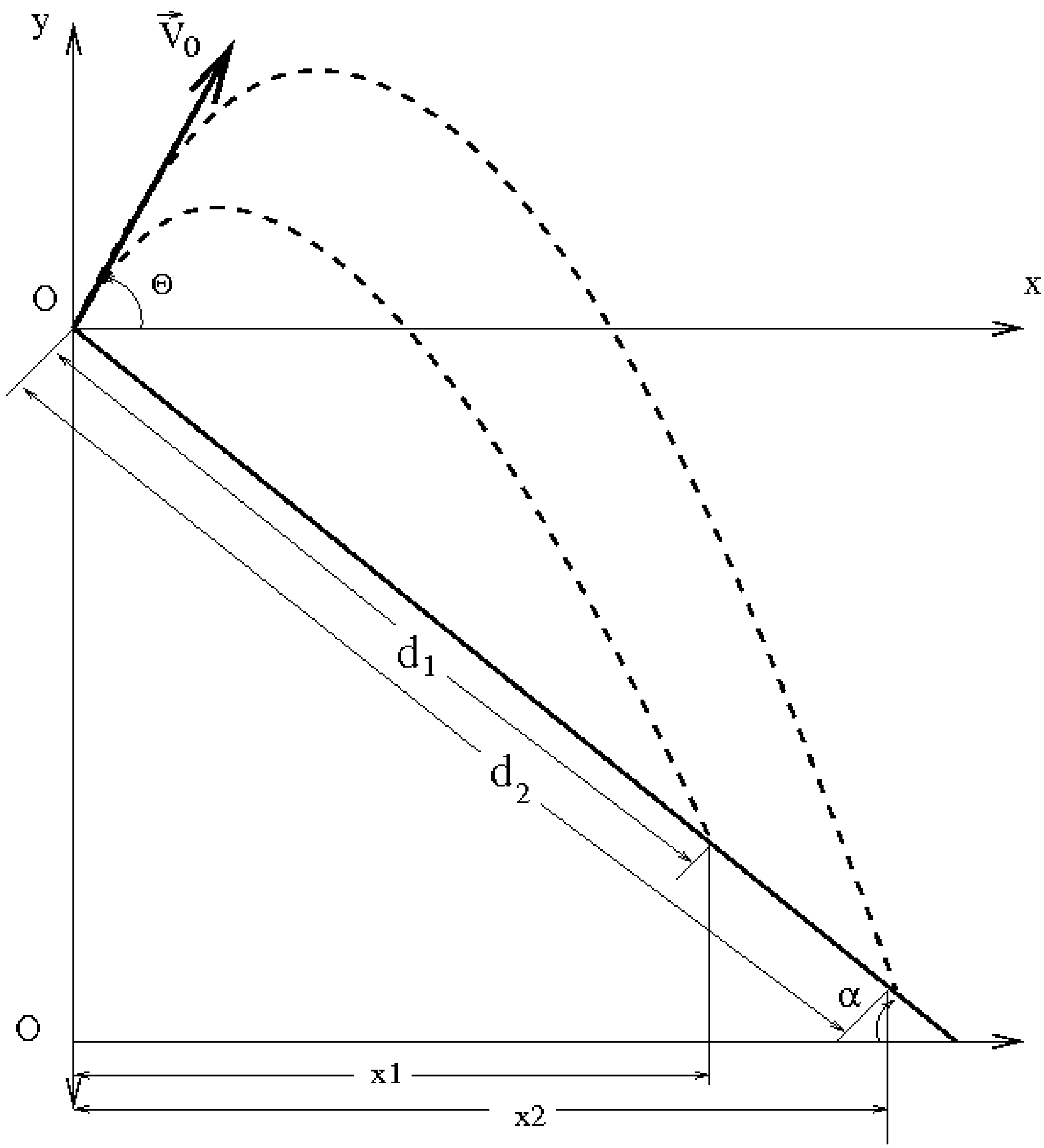
$$|\vec{v}_c| = \sqrt{4t^2 + 16 + \frac{1}{25}\left(t^2 - 4t + 4\right)} = \frac{1}{5}\sqrt{101t^2 - 4t + 404}$$

We can show that for the time interval (0-0.80) the speed of the fly shadow is always positive:

$$t = 0 \Rightarrow |\vec{v}_c| = \frac{1}{5}\sqrt{404} = 4.02$$

$$t = 0.80 \Rightarrow |\vec{v}_c| = \frac{1}{5}\sqrt{465.44} = 4.31$$

Problem # 3



$$y_1 = (\tan\theta)x_1 - \left(\frac{g}{2v_{01}^2\cos^2\theta}\right)x_1^2$$

$$x_1 = d_1\cos\alpha$$

$$d_1 = \sqrt{x_1^2 + y_1^2} \Rightarrow y_1 = \pm\sqrt{d_1^2 - x_1^2} = \pm d_1\sqrt{1-\cos^2\alpha} = \pm d_1\sin\alpha$$

But $\alpha = -60^\circ \rightarrow |\alpha| = 60^\circ$ and from the graph we have:
$y_1 = -d_1\sin(|\alpha|) = d_1\sin\alpha$. Now solve for v_{01}:

$$d_1\sin\alpha = (\tan\theta)d_1\cos\alpha - \left(\frac{g}{2v_{01}^2\cos^2\theta}\right)d_1^2\cos^2\alpha$$

$$v_{01}^2 = \frac{1}{2}gd_1\left(\frac{\cos\alpha}{\cos^2\theta}\right)(\tan\theta - \tan\alpha)^{-1}$$

$$v_{01} = \sqrt{\frac{gd_1}{2}}\frac{1}{\cos\theta}\sqrt{\frac{\cos\alpha}{(\tan\theta - \tan\alpha)}}$$

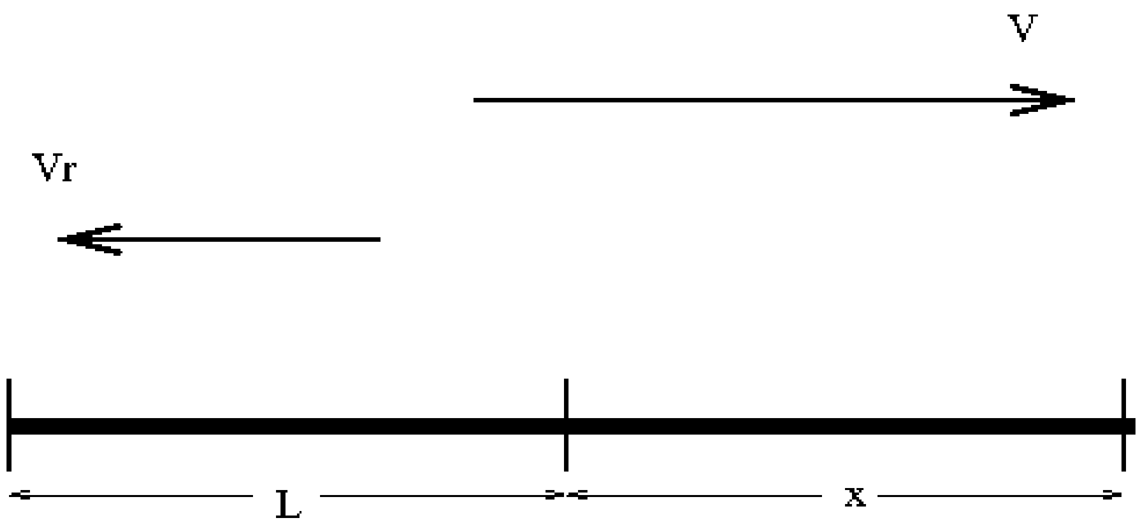
and the same for v_{02}

$$v_{02} = \sqrt{\frac{gd_2}{2}}\frac{1}{\cos\theta}\sqrt{\frac{\cos\alpha}{(\tan\theta - \tan\alpha)}}$$

$$\therefore \vartriangle v_0 = v_{02} - v_{01} = \sqrt{\frac{g}{2}}\left(\sqrt{d_2} - \sqrt{d_1}\right)\frac{1}{\cos\theta}\sqrt{\frac{\cos\alpha}{(\tan\theta - \tan\alpha)}} = 0.29 \text{ m/s}$$

$$\vartriangle v_0 = v_{02} - v_{01} \approx 0.29 \text{ m/s}$$

Problem # 4



If we work in the reference frame of the river bank (fixed Earth) we have the system:

$$\begin{cases} x = (v - v_r)\vartriangle t_1 \\ L + x = (v + v_r)\vartriangle t_2 \\ L = v_r(\vartriangle t_1 + \vartriangle t_2) \end{cases}$$

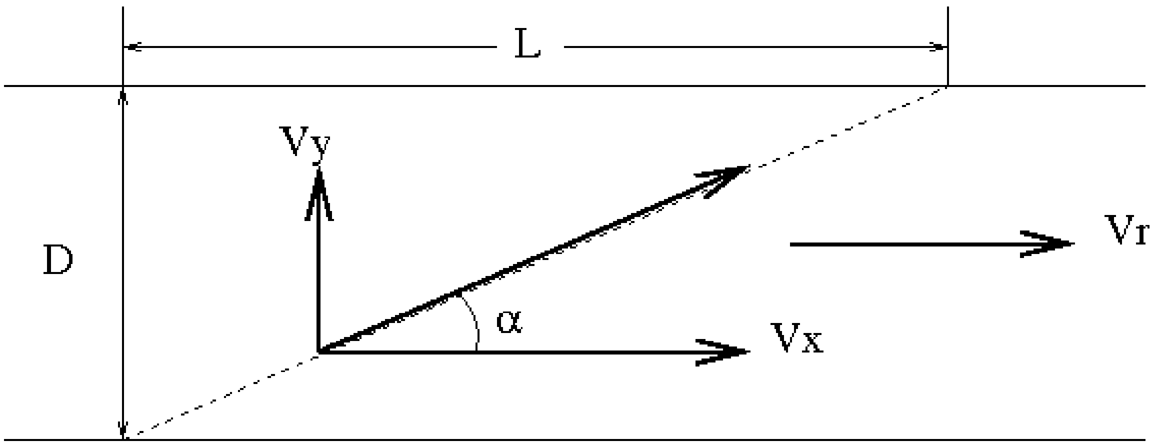
$$\begin{cases} L = (v + v_r)\vartriangle t_2 - (v - v_r)\vartriangle t_1 \\ L = v_r(\vartriangle t_1 + \vartriangle t_2) \end{cases}$$

$$\therefore L = (v + v_r)(\frac{L}{v_r} - \vartriangle t_1) - (v - v_r)\vartriangle t_1 \Rightarrow L = 2v_r \vartriangle t_1$$

$$\therefore v_r = \frac{1}{2}\frac{L}{\vartriangle t_1} = 2 \text{ km/h}$$

The final formula can be obtained directly if we work in the reference frame of the river.

Then immediately the time to paddle up-river is equal to the time to paddle down-river,
i.e. $\Delta t_1 = \Delta t_2 \Rightarrow L = v_r(\Delta t_1 + \Delta t_2) = 2\Delta t_1 v_r$.



$$|\vec{v}| = 6 \text{ km/h}, D = 100 \text{ m}, \alpha = 30^\circ$$

$$D = v_y \bullet \Delta t \Rightarrow \Delta t = \frac{D}{v \sin\alpha} = 120 \text{ s} = 2 \text{ min}$$

$$L = (v_x + v_r)\Delta t = \left[v\cos(30^\circ) + v_r \right]\Delta t = 240 \text{ m}$$