PHY 180F
SOLUTIONS TO ASSIGNMENT 2 2005

1) After the mass is hung on the spring:

$kX = mg$

$X = \frac{mg}{k}$

If down is positive, the total force on the mass is:

$F = -kx + mg \quad (1)$

Newton's second law is:

$F = ma = m\frac{d^2x}{d^2t} \quad (2)$

Equating (1) and (2)

$m\frac{d^2x}{dt^2} = -kx + mg$

But: $x = x' + X$ and therefore

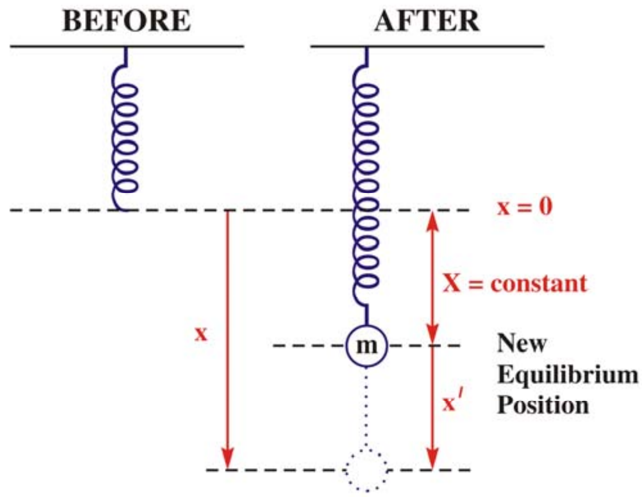
$m\frac{d^2(x'+X)}{dt^2} = -k(x'+X) + mg$

$m\frac{d^2x'}{dt^2} + 0 = -kx' - kX + mg$

But $X = \frac{mg}{k}$ and therefore

$m\frac{d^2x'}{dt^2} = -kx' - k\frac{mg}{k} + mg$

$m\frac{d^2x'}{dt^2} = -kx'$



But this is the same as the equation $m\frac{d^2x}{dt^2} = -kx$ that we had in class except that x' replaces x and we know that the solution to this equation is simple harmonic motion of the form $x = A\sin(\omega t + \phi)$.

Thus the solution to $m\frac{d^2x'}{dt^2} = -kx'$ is $x' = A\sin(\omega t + \phi)$ where $\omega = \sqrt{\frac{k}{m}}$ and this means that the mass executes simple harmonic motion about the new equilibrium position since x' is the displacement from the new equilibrium position.

2) YES you are justified in throwing out measurement 3.
Consider all the trials except 3. The average is 31.69 seconds and the standard deviation is 0.20 seconds. The question is whether or not measurement 3 comes from the same sample of measurements as all the other measurements except 3.

Measurement 3 is (33.91 – 31.69)/0.20 or 11 standard deviations away from the mean of the nine measurement excluding 3. The probability that 3 could belong the distribution that the other 9 came from is exceedingly small (recall that the probability is 1% for 3 standard deviations).

It is incorrect to use the standard deviation calculated using all 10 measurements. If you do this then measurement 3 is only 3 standard deviations from the mean and there is no good reason to reject it and so you might think that it should be kept. However, calculating the standard deviation including 3 means that you explicitly are saying the 3 does belong to the distribution and so the large standard deviation reflects this assumption. This is known as the “Emperor of China” proof and it goes like this.

a) Assume that I am the Emperor of China.
b) Next we have several lines of mumbo jumbo and pages of calculations.
c) Finally the calculations “prove” what we assumed in the first place and the conclusion is:
d) Therefore I am the Emperor of China. QED!

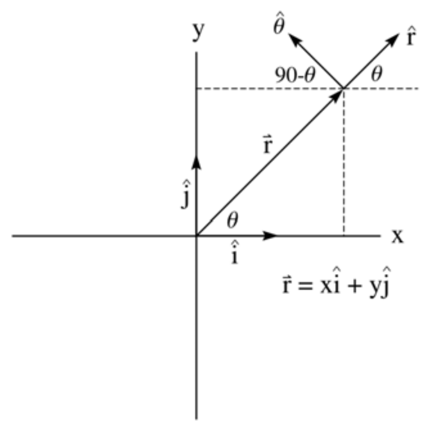
3a)

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$ **or**

$$\hat{r} = \cos\theta \cdot \hat{i} + \sin\theta \cdot \hat{j} = \frac{x}{r} \cdot \hat{i} + \frac{y}{r} \cdot \hat{j} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\begin{aligned}\hat{\theta} &= -\cos(90^\circ - \theta) \cdot \hat{i} + \cos\theta \cdot \hat{j} \\ &= -\sin\theta \cdot \hat{i} + \cos\theta \cdot \hat{j} \\ &= -\frac{y}{r} \cdot \hat{i} + \frac{x}{r} \cdot \hat{j}\end{aligned}$$

$$\therefore \hat{\theta} = \frac{-y \cdot \hat{i} + x \cdot \hat{j}}{\sqrt{x^2 + y^2}}$$



3b)

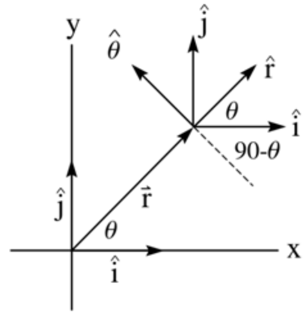
$$\begin{aligned}\vec{A} &= \sin\theta \cdot \hat{r} - r \cdot \cos\theta \cdot \hat{\theta} \\ &= \frac{y}{\sqrt{x^2 + y^2}}\left(\frac{x \cdot \hat{i} + y \cdot \hat{j}}{\sqrt{x^2 + y^2}}\right) - \sqrt{x^2 + y^2}\,\frac{x}{\sqrt{x^2 + y^2}}\left(\frac{-y \cdot \hat{i} + x \cdot \hat{j}}{\sqrt{x^2 + y^2}}\right) \\ &= \frac{yx}{x^2 + y^2} \cdot \hat{i} + \frac{y^2}{x^2 + y^2} \cdot \hat{j} + \frac{xy}{x^2 + y^2} \cdot \hat{i} - \frac{x^2}{x^2 + y^2} \cdot \hat{j}\end{aligned}$$

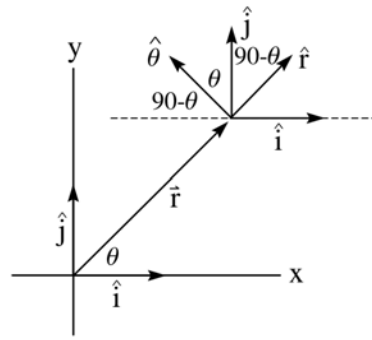
$$\therefore \vec{A} = \left(\frac{xy}{x^2 + y^2} + \frac{xy}{\sqrt{x^2 + y^2}}\right) \cdot \hat{i} + \left(\frac{y^2}{x^2 + y^2} - \frac{x^2}{\sqrt{x^2 + y^2}}\right) \cdot \hat{j}$$

3c)

$$\hat{i} = \cos\theta \cdot \hat{r} - \cos(90^\circ - \theta) \cdot \hat{\theta}$$
$$= \cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta}$$

$$\hat{j} = \cos(90 - \theta) \cdot \hat{r} + \cos\theta \cdot \hat{\theta}$$
$$= \sin\theta \cdot \hat{r} + \cos\theta \cdot \hat{\theta}$$

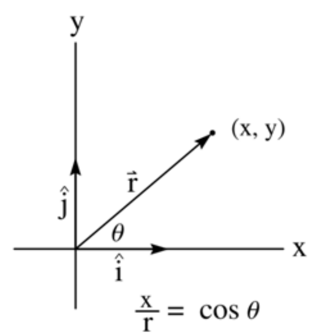




3d)

$$\vec{B} = xy^2 \cdot \hat{i} + x^2 y \cdot \hat{j}$$

$$\vec{B} = r\cos\theta \cdot r^2 \sin^2\theta(\cos\theta \cdot \hat{r} - \sin\theta \cdot \hat{\theta}) +$$
$$r^2\cos^2\theta \cdot r\sin\theta(\sin\theta \cdot \hat{r} + \cos\theta \cdot \hat{\theta})$$
$$= r^3\cos^2\theta\sin^2\theta \cdot \hat{r} - r^3\cos\theta\sin^3\theta \cdot \hat{\theta} +$$
$$r^3\cos^2\theta\sin^2\theta \cdot \hat{r} + r^3\sin\theta\cos^3\theta \cdot \hat{\theta}$$
$$= r^3\left[2\sin^2\theta\cos^2\theta \cdot \hat{r} + (\sin\theta\cos^3\theta - \cos\theta\sin^3\theta) \cdot \hat{\theta}\right]$$



4) This problem is NOT as easy as it sounds unless you adopt the language that was used in class which assumes that the water starts to "fall" as soon as it leaves the fishes mouth. If you assume that "fall" means that the water has to descend a certain amount from its highest point then you get a much more difficult problem which we solve first.

Expressing the vertical displacement as a function of the horizontal one

$$y = (\tan\theta_i)x - \left(\frac{g}{2\upsilon_i^2\cos^2\theta_i}\right)x^2$$

(equation derived in the textbook), and also using the relation for maximum height of the projectile (eq. 4.13 in the textbook)

$$h = \frac{\upsilon_i^2 \sin^2\theta_i}{2g},$$

we can specify for this particular problem

$y \equiv H$, $x \equiv L_x = L_0\cos\theta_0$ and h $\to$ H+h.

Here $L_0 = 2.00$ m, $\theta_0 = 30.0^\circ$, **so that**

$L_x = L_0 \cos\theta_0 = 1.73$ m, $H = L_0 \sin\theta_0 = 1$ m, $h = 3.00$ cm=0.03 m.

We can interpret the problem to mean that the displacement from fish to bug is

$\vec{r} = L_x \cdot \hat{i} + H \cdot \hat{j} = (1.73 \text{ m})\hat{i} + (1.00 \text{ m})\hat{j}$.

This should be the end point of the projectile motion. Using the above information we can construct the following (non-linear!) algebraic system of 2 equations:

$$\begin{cases} H = (\tan\theta_i)L_x - \dfrac{1}{2}\dfrac{gL_x^2}{\upsilon_i^2 \cos^2\theta_i} \\ (H+h) = \dfrac{\upsilon_i^2 \sin^2\theta_i}{2g} \end{cases}$$

for the 2 unknowns θ_i and υ_i. **Now we have to solve this algebraic system.**

The easiest way is to express υ_i^2 **from the second equation, and plug it in the first one, and the detail procedure comes about as:**

$$\upsilon_i^2 = \frac{2g(H+h)}{\sin^2\theta_i} \text{ (from eq. 2 in the textbook)}$$

Then

$$H = (\tan\theta_i)L_x - \frac{1}{2}\frac{\cancel{g}L_x^2}{\dfrac{2\cancel{g}(H+h)}{\sin^2\theta_i}\cos^2\theta_i}$$

$$H = (\tan\theta_i)L_x - \frac{1}{4}\frac{L_x^2}{(H+h)}\left(\frac{\sin^2\theta_i}{\cos^2\theta_i}\right)$$

$$H = (\tan\theta_i)L_x - \frac{1}{4}\frac{L_x^2}{(H+h)}(\tan\theta_i)^2$$

and setting $q \equiv (\tan\theta_i)$ **we end up with the following quadratic equation**

$$H = L_x \cdot q - \frac{1}{4}\frac{L_x^2}{(H+h)} \cdot q^2,$$

$$\frac{4H \cdot (H+h)}{L_x^2} = \frac{4L_x \cdot (H+h)}{L_x^2} q - q^2,$$

$$\therefore q^2 - 4\frac{(H+h)}{L_x} q + 4\frac{H(H+h)}{L_x^2} = 0.$$

with the following solution

$$q_{1,2} = \frac{2(H+h)}{L_x} \pm \sqrt{\left[\frac{2(H+h)}{L_x}\right]^2 - 4\frac{H(H+h)}{L_x^2}}$$

$$= \frac{2(H+h)}{L_x} \pm \frac{2(H+h)}{L_x}\sqrt{1-\left(\frac{H}{H+h}\right)}$$

$$= \frac{2(H+h)}{L_x}\left[1 \pm \sqrt{1-\left(\frac{H}{H+h}\right)}\right]$$

$$= \frac{2(1.00\text{m}+0.03\text{m})}{1.73\text{m}}\left[1 \pm \sqrt{1-\left(\frac{1.00\text{m}}{1.03\text{m}}\right)}\right]$$

$$= \frac{2.06}{1.73}\left[1 \pm \sqrt{1-0.971}\right] = 1.191\left[1 \pm 0.170\right]$$

$$\therefore\ q_1 = q_+ = 1.393, \quad q_2 = q_- = 0.989$$

$$\theta_{i,1} = \theta_{i,+} = \tan^{-1}(1.393) = 54.33^\circ,$$

$$\theta_{i,1} = \theta_{i,+} = \tan^{-1}(0.989) = 44.67^\circ$$

Observe that there are two physically possible solutions: one above and one below 45.00°. The first solution tells you that the bug is being stricken on the downward motion of the projectile trajectory, while for the second solution the bug is intercepted on the upward motion on the projectile trajectory. Any way, the possible solutions for the direction of the initial velocity are

$$\theta_{i,-} \le \theta_i \le \theta_{i,+} \ \rightarrow\ 44.67^\circ \le \theta_i \le 54.33^\circ$$

For the magnitude of the initial velocity we get respectively:

$$\upsilon_i = \frac{\sqrt{2g(H+h)}}{\sin\theta_i} = \frac{\sqrt{2\cdot 9.80\cdot 1.03}}{\sin\theta_i} = \frac{4.493}{\sin\theta_i}$$

$$\upsilon_{i,1} = \upsilon_{i,+} = \frac{4.493}{\sin(54.33^\circ)} = 5.531\text{ m/s}$$

$$\upsilon_{i,2} = \upsilon_{i,-} = \frac{4.493}{\sin(44.67^\circ)} = 6.391\text{ m/s}$$

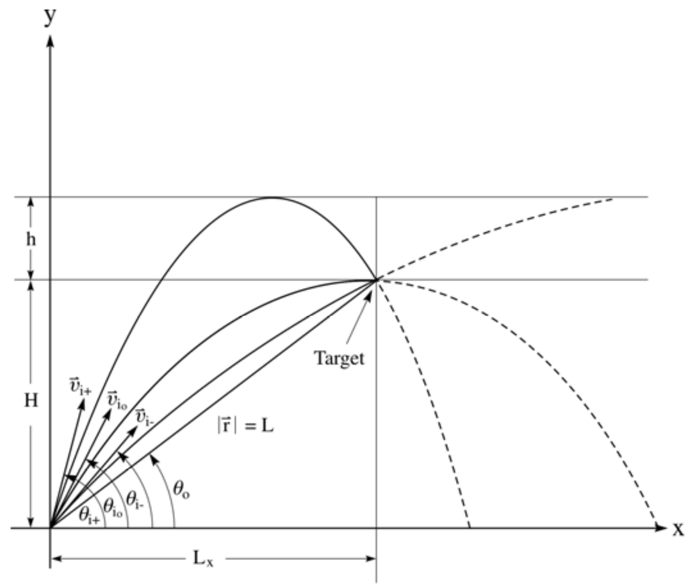
Observe that when $h \rightarrow 0$, we end up with only one possible solution (which is actually the maximum height!)

$$q_0 = \frac{2H}{L_x} = \frac{2 \cdot 1.00\text{m}}{1.73\text{m}} = 1.156,$$

$$\theta_{i,0} = \tan^{-1}(1.156) = 49.14^\circ,$$

$$\upsilon_i = \frac{\sqrt{2gH}}{\sin\theta_i} = \frac{\sqrt{2 \cdot 9.80 \cdot 1.00}}{\sin(49.14^\circ)} = \frac{4.427}{0.756} = 5.853 \text{ m/s}$$

Here you can see graphs of the possible solutions:



$L_x = L\cos\theta_o$

A more simple approach is to use the language that we used in class and assume that the water should drop 0.03 m during its flight.
The fish must aim at a point 0.03 m above the bug. This assumption forces all the possible trajectories to lie below this point.

Then the initial velocity of the water is directed through the point with displacement

$$L_x\hat{i} + (H+h)\hat{j} = (1.73\text{m})\hat{i} + (1.03\text{m})\hat{j} = 2.012\text{m at } 30.7^\circ$$

This fixes the direction of the projectile motion, which at the end will lead to only one possible solution. For the time of flight of a water droplet

$$L_x = \upsilon_i \cos\theta_i \cdot t \;\rightarrow\; t = \frac{L_x}{\upsilon_i \cos\theta_i}$$

The vertical motion is given by

$$y_f - y_i = \upsilon_{yi}t + \frac{1}{2}a_y t^2 \Rightarrow -h = -\frac{1}{2}g\left(\frac{L_x}{\upsilon_i\cos\theta_i}\right)^2$$

which describes the motion of the water droplet from the moment after it reached its maximum height (free fall with zero initial velocity), but with the WRONG time scale!

Now solving for υ_i they finally get

$$\upsilon_i = \frac{L_x}{\cos\theta_i}\sqrt{\frac{g}{2h}} = \frac{1.73\text{m}}{\cos(30.7^\circ)}\sqrt{\frac{9.80\text{m/s}^2}{2\times 0.03}} =$$

$$= 2.012\text{m}(12.78\text{s}^{-1}) = 25.7\text{m/s}$$

As you can see, there is a vast difference in the magnitude of the initial velocity in this solution due to the time scale for the free fall of the water droplet.

Finally, if we assume the fixed initial velocity direction of 30.7° **and use our first equation for the vertical displacement as a function of the horizontal one (which is true for every point on the trajectory) we get**

$$\upsilon_i = \frac{1}{\cos\theta_i}\sqrt{\frac{gL_x}{2\left(\tan\theta_i - \frac{H}{L_x}\right)}}$$

$$= \frac{1}{\cos(30.7^\circ)}\sqrt{\frac{9.80\mathrm{m/s}^2 \times 1.73\mathrm{m}}{2\left[\tan(30.7^\circ) - \frac{1\mathrm{m}}{1.73\mathrm{m}}\right]}} = 27.00\mathrm{m/s}$$

Now you can appreciate that the same physical problem can have multitude of solutions depending on the approach taken and the assumptions that you make.