PHY180H1F
Solutions to problem set #5, due October 28, 2004

#1 (7.56)
a) The initial energies stored in spring 1 and 2 are:

$$E_{i1} = \frac{1}{2}k_1 x_{i1}^{\ 2}$$

$$E_{i2} = \frac{1}{2}k_2 x_{i2}^{\ 2}$$

If we apply a force such that spring 1 is further stretched to the right by x_a its total energy will increase by the work done. The energy of spring is now:

$$\begin{aligned} E_1 &= \frac{1}{2}k_1 (x_{i1} + x_a)^2 \\ &= \frac{1}{2}k_1 \left(x_{i1}^{\ 2} + 2x_a x_{i1} + x_a^{\ 2}\right) \\ &= \frac{1}{2}k_1 x_{i1}^{\ 2} + \frac{1}{2}k_1\left(2x_a x_{i1} + x_a^{\ 2}\right) \\ &= E_{i1} + W_1 \end{aligned}$$

Therefore:

$$\boxed{W_1 = \frac{1}{2}k_1\left(2x_a x_{i1} + x_a^{\ 2}\right)}$$

b) Since in a) spring 1 is further stretched by xa, spring 2 must be compressed by xa, therefore the total work done on spring 2 will decrease the energy initially stored by spring 2, that is:

$$\begin{aligned} E_2 &= \frac{1}{2}k_2 (x_{i2} - x_a)^2 \\ &= \frac{1}{2}k_2 \left(x_{i2}^{\ 2} - 2x_a x_{i2} + x_a^{\ 2}\right) \\ &= \frac{1}{2}k_2 x_{i2}^{\ 2} + \frac{1}{2}k_2\left(-2x_a x_{i2} + x_a^{\ 2}\right) \\ &= E_{i2} + W_2 \end{aligned}$$

Therefore:

$$\boxed{W_2 = \frac{1}{2}k_2\left(x_a^{\ 2} - 2x_a x_{i2}\right)}$$

c) Initially the given system of springs is in equilibrium and therefore the forces exerted by the springs must be in balance, that is:

$$\sum F = 0$$

$$F_1 - F_2 = 0$$

$$\frac{1}{2}k_1 x_{i1} = \frac{1}{2}k_2 x_{i2}$$

$$x_{i2} = \frac{k_1}{k_2} x_{i1}$$

d) The total work done by the force applied is:

$$W_T = W_1 + W_2$$

$$= \frac{1}{2}k_1\left(x_a{}^2 + 2x_a x_{i1}\right) + \frac{1}{2}k_2\left(x_a{}^2 - 2x_a x_{i2}\right)$$

$$= \frac{1}{2}(k_1 + k_2)x_a{}^2 + (k_1 x_{i1} - k_2 x_{i2})x_a$$

Using result from part c) we get:

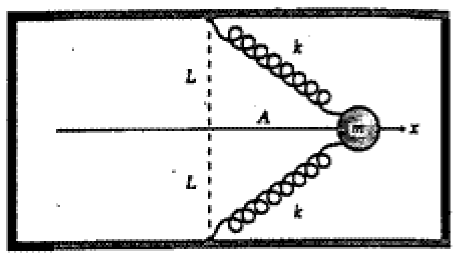
$$\boxed{W_T = \frac{1}{2}(k_1 + k_2)x_a{}^2}$$

#2 (7.58)

a) The new length of each spring is $\sqrt{x^2 + L^2}$, so its extension is $\sqrt{x^2 + L^2} - L$ and the force it exerts is $k(\sqrt{x^2 + L^2} - L)$ toward its fixed end. The y components of the two spring forces add to zero. Their x components add to

$$\mathbf{F} = -2\mathbf{i}k(\sqrt{x^2 + L^2} - L)x/\sqrt{x^2 + L^2}$$

$$\mathbf{F} = \boxed{-2kx\mathbf{i}\,(1 - L/\sqrt{x^2 + L^2})}$$



(top view)

b) $W = \int_i^f F_x\, dx$

$$W = \int_A^0 -2kx\,(1 - L/\sqrt{x^2 + L^2})\,dx$$

$$W = -2k\int_A^0 x\, dx + kL\int_A^0 (x^2 + L^2)^{-1/2}\, 2x\, dx$$

$$W = -2k\frac{x^2}{2}\Bigg|_A^0 + kL\frac{(x^2 + L^2)^{1/2}}{(1/2)}\Bigg|_A^0$$

$$W = -0 + kA^2 + 2kL^2 - 2kL\sqrt{A^2 + L^2}$$

$$W = \boxed{2kL^2 + kA^2 - 2kL\sqrt{A^2 + L^2}}$$

#3 (7.60)

a) To express the given vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ in unit vector notation in Cartesian coordinates, we obtain the individual x and y vector components and sum them together:

$$\mathbf{F}_1 = 25.0\left(\cos 35.0^\circ\,\hat{i} + \sin 35.0^\circ\,\hat{j}\right) = \left(20.5\hat{i} + 14.3\hat{j}\right) \quad N$$

$$\mathbf{F}_2 = 42.0\left(\cos 150.0^\circ\,\hat{i} + \sin 150.0^\circ\,\hat{j}\right) = \left(-36.4\hat{i} + 21.0\hat{j}\right) \quad N$$

b) The total force on the object is:

$$\mathbf{F}_T = \mathbf{F}_1 + \mathbf{F}_2$$

$$= \left(20.5\hat{i} + 14.3\hat{j}\right) + \left(-36.4\hat{i} + 21.0\hat{j}\right)$$

$$\mathbf{F}_T = \left(-15.9\hat{i} + 35.3\hat{j}\right) \quad N$$

c) Using Newton's 2nd law, the acceleration is:

$$\mathbf{a} = \frac{\mathbf{F}_T}{m} = \frac{1}{5.00}\left(-15.9\hat{i} + 35.3\hat{j}\right)$$

$$= \left(-3.18\hat{i} + 7.07\hat{j}\right) \quad m/s$$

d) We use a kinematics equation to obtain the velocity of the object at t = 3.00 seconds

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$

$$= \left(4.00\hat{i} + 2.50\hat{j}\right) + \left(-3.18\hat{i} + 7.07\hat{j}\right)(3.00)$$

$$\mathbf{v}_f = \left(-5.54\hat{i} + 23.7\hat{j}\right) \quad m/s$$

e) The objects location (displacement) at t = 3.00 seconds is also given by a kinematics equation

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2$$

$$\Delta\mathbf{r} = \left(4.00\hat{i} + 2.50\hat{j}\right)(3.00) + \frac{1}{2}\left(-3.18\hat{i} + 7.07\hat{j}\right)(3.00)^2$$

$$\Delta\mathbf{r} = \left(-2.30\hat{i} + 39.3\hat{j}\right) \quad m$$

f) The final kinetic energy at t = 3.00 seconds of the object is:

$$K_f = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\mathbf{v}_f \cdot \mathbf{v}_f\right)$$

$$= \frac{1}{2}(5.00)\left(5.54^2 + 23.7^2\right)$$

$$K_f = 1.48 \quad kJ$$

g) The final kinetic energy using

$$
\begin{aligned}
K_f &= \frac{1}{2} m v_i^{\,2} + \sum \mathbf{F} \cdot \Delta \mathbf{r} \\
&= \frac{1}{2}(5.00)\left(4.00^2 + 2.50^2\right) + \left(-15.9\hat{i} + 35.3\hat{j}\right) \cdot \left(-2.30\hat{i} + 39.3\hat{j}\right) \\
&= 2.50\,(16.0 + 6.25) + \left(15.9\,(2.30) + 35.3(39.3)\right) \\
&= 1.48 \quad kJ
\end{aligned}
$$

#4 (7.68)

a) The power available is

$$
\begin{aligned}
P &= \frac{1}{2} D \rho \pi r^2 v^3 \\
&= \frac{1}{2}(1.20)\pi(1.50)^2(8.00)^3 \\
&= 2171 \quad \mathrm{J}/s \\
P &= 2.17\ k\mathrm{W}
\end{aligned}
$$

b) Similarly,

$$
\begin{aligned}
P &= \frac{1}{2} D \rho \pi r^2 v^3 \\
&= \frac{1}{2}(1.20)\pi(1.50)^2(24.0)^3 \\
&= 58629 \quad \mathrm{J}/s \\
P &= 58.6\ k\mathrm{W}
\end{aligned}
$$