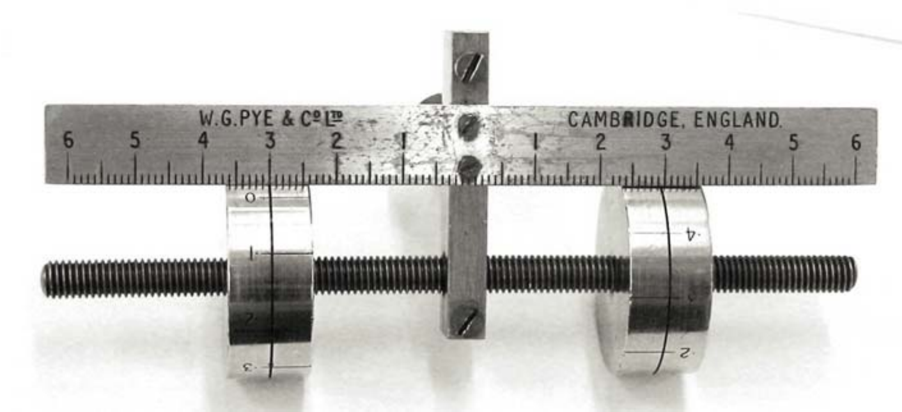
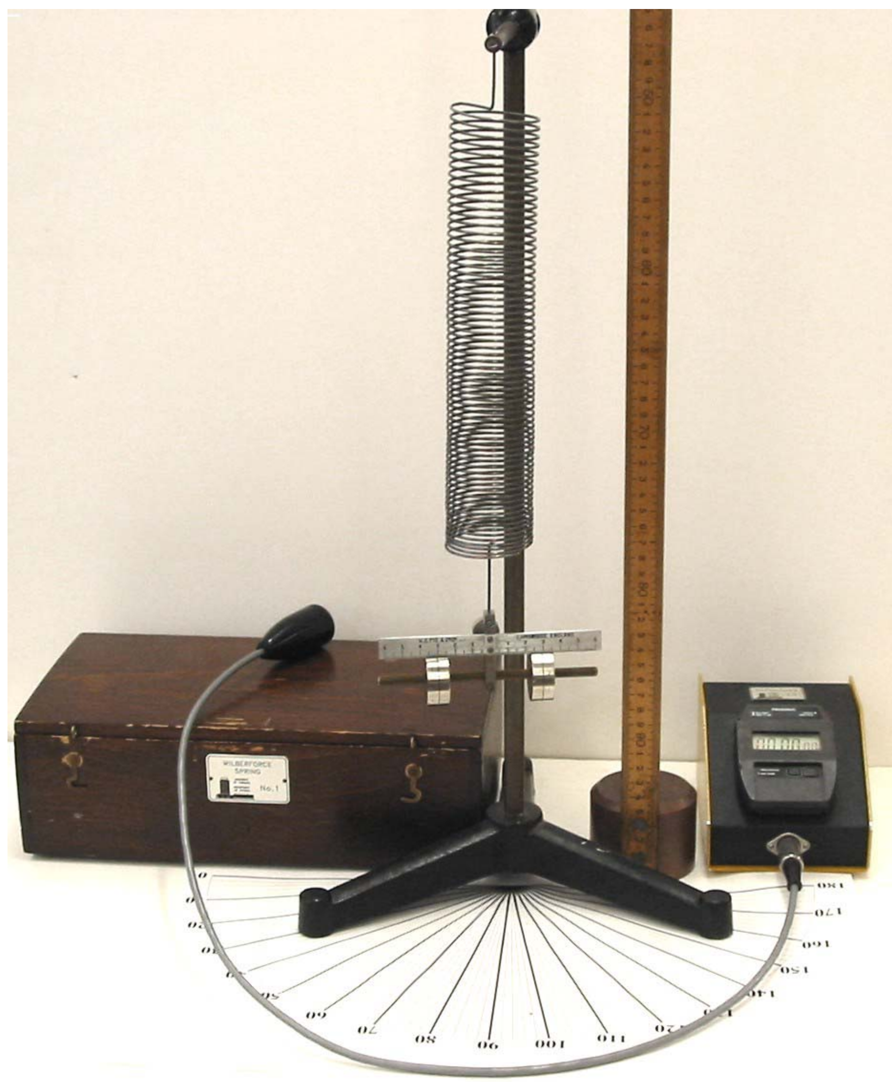
THE WILBERFORCE PENDULUM

For a **1** weight experiment do **Part 1**. For a **2** weight experiment do **Part1** and **Part 2.**

INTRODUCTION





The Wilberforce pendulum (also known as a Wilberforce spring) is a spectacular example of a system in which coupling between types of oscillation results in complete transfer of energy between translational and rotational harmonic motion. This means that at one point in time the motion can be up and down translational motion without any rotation of the apparatus and at a later time the motion may be entirely rotational without any up and down movement. One can observe the periodic interchange of energy between the two types of oscillation. An interactive animation of the interchange of energy in a system of coupled oscillators will help you appreciate this phenomenon.

A complete description of the Wilberforce pendulum is given in R.E. Berg and T.S. Marshall, *Wilberforce Pendulum Oscillations and Normal Modes*, Am. J. Phys., **59**, 32-38 (1991). It is not necessary to read this reference since it may be difficult for a first year student but it would be useful if a complete mathematical description is desired.

The simple treatment of a mass on the end of a spring that you learn in lectures says that when the mass is oscillating up and down the period should depend only on the particular spring used and the total mass on the end of the spring. The period for this massless spring is not applicable for the Wilberforce Spring: *i.e.*,

$$\mathbf{T_{trans}} \neq 2\pi\sqrt{\frac{\mathbf{M}}{\mathbf{k}}} \qquad (1)$$

where **M** = mass of the bob, and **k** = spring constant (force/extension or **F/x**), because in this situation you cannot neglect the fact that the centre of mass of the spring is also oscillating. In fact, for the uniform massy spring:

$$\mathbf{T_{trans}} = 2\pi\sqrt{\frac{\mathbf{M} + \frac{\mathbf{M_{sp}}}{3}}{\mathbf{k}}} \qquad (2)$$

where $\mathbf{M_{sp}}$ = mass of the spring.

When the pendulum is undergoing rotational motion the period of the rotational oscillation depends on the particular spring and on the moment of inertia **I** of the mass (moveable masses plus frame system) on the end of the spring. Analogous to equation (2) for the massy spring with the frame and masses attached to the end,

$$\mathbf{T_{rot}} = 2\pi\sqrt{\frac{\mathbf{I} + \frac{\mathbf{I_{sp}}}{3}}{\kappa}} \qquad (3)$$

where $\mathbf{I_{sp}}$ = moment of inertia of the spring and $\boldsymbol{\kappa}$ = torsion constant of the spring (torque/angular displacement or $\boldsymbol{\tau/\theta}$). The moment of inertia of the mass is $\mathbf{I = 2md^2 + I_f + 2I_m}$, where **m** is the mass of a moveable mass, **d** is the distance of each of the moveable masses from the centre, $\mathbf{I_f}$ is the moment of inertia of the frame and $\mathbf{I_m}$ comes from the parallel axis theorem and is a result of the fact that the masses are not point masses. Since $\mathbf{I_f}$, $\mathbf{I_m}$ $\mathbf{I_{sp}}$ are constants they can all be lumped into one constant $\mathbf{I_0}$.

Equation (3) becomes: $$\mathbf{T_{rot}} = 2\pi\sqrt{\frac{\mathbf{I_0 + 2md^2}}{\kappa}} \qquad (4)$$

In this system, these two types of harmonic motion, translational and rotational, are not entirely independent; there is a slight coupling between them. This results from the fact that the spring has a slight tendency to coil and uncoil as it is extended or compressed. The Wilberforce spring is thus an example of two weakly coupled resonant systems, other examples being the splitting of energy levels in the ammonia molecule, two simple pendulums of similar length with a spring joining the upper parts of their strings (see the animation in the first reference). Thus, the Wilberforce spring is a good way to study mechanical resonance in coupled systems.

THE EXPERIMENT

Part 1

Adjusting for Balance

The metal scale is not located under the geometrical centre of the spring. If the centre of mass is not under the geometrical centre then as the mass rotates it will also start to swing. There is a cylindrical counterweight which allows you to compensate for this imbalance.

a) Set the mass rotating and adjust the position of the counterweight to minimize swinging.
b) Determine and record the distance of the counterweight from the end in case it is moved by other students before you do Part 2.

Determining Resonance

Resonance occurs when $\mathbf{T_{rot}}$ is equal to $\mathbf{T_{trans}}$ (if there were no coupling) and so you will first want to determine this condition. Note that the apparatus allows you to vary $\mathbf{T_{rot}}$, but not $\mathbf{T_{trans}}$.

a) Measure $\mathbf{T_{trans}}$.
b) Referring to equation (4), make a hand plot of $\mathbf{T^2_{rot}}$ versus $\mathbf{d^2}$ for about 5 values of $\mathbf{d}$ and by interpolation determine the value of $\mathbf{d_r}$ for resonance.
c) Note that this calculated value for $\mathbf{d_r}$ may only be approximate and so with this value you may be only approaching resonance. As you approach resonance you will notice that for a while the amplitude of the translational motion will decrease and the amplitude of the rotational motion will increase and then a short time later the reverse process will occur. The closer you get to resonance the larger will be the period for the interchange of energy for the two types of oscillation. Start with your calculated approximation for $\mathbf{d_r}$ and experimentally determine resonance exactly by making small adjustments of $\mathbf{d}$, i.e. within the approximate range $\mathbf{d_r}$ - 1mm $\leq \mathbf{d} \leq \mathbf{d_r}$ + 1mm. At resonance, at one point in time the motion will be entirely translational without any rotation of the apparatus and at a later time the motion will be entirely rotational without any up and down movement. Also, at resonance the amplitudes of both rotation and translation will alternately be at a maximum.

Energy Transfer at Resonance

The energy of the translational motion can be determined from the linear amplitude and the rotational energy from the angular amplitude if the force constant **k** and the torsion constant $\boldsymbol{\kappa}$ are known.

a) Measure the maximum linear and rotational amplitudes at resonance. Note that these measurements, and hence the energy calculations, are crude because of parallax (e.g. when looking down from above the rotating frame onto the protractor).
b) From a plot of $\mathbf{T^2_{rot}}$ versus $\mathbf{d^2}$ and determine the torsion constant $\boldsymbol{\kappa}$ using equation (4).
c) Determine the force constant **k** by hanging the provided masses from the end of the spring with the special hook and pointer.
d) Calculate the maximum translational and rotational energies and compare.

Part 2

Some additional theory concerning normal modes is required for part 2. A complete but rather long (13 pages) explanation of normal modes is given in A.P. French, *Vibrations and Waves*, Chap. 5. Here is a much shorter explanation which should be sufficient.

The motion that the Wilberforce pendulum experiences at any given time is a combination of both translational and rotational motion. Just as the position of an object can be treated as a vector i.e. **(x, y)** with vertical and horizontal components, we can represent a given state of motion of the Wilberforce pendulum as the vector $(\mathbf{z}, \boldsymbol{\theta})$ with linear (vertical) and angular components. It turns out that **any** position that the pendulum might occupy during its motion is created by a linear combination of two such vectors. That is: $(\mathbf{z},\boldsymbol{\theta}) = \mathbf{a}\,\mathbf{v}_1 + \mathbf{b}\,\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ both have vertical and angular components. Both of these vectors have their own frequency. At any given time, the position of the pendulum may be a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$ alone or it may be a mixture of both. When the motion of the oscillator is just due to one of these vectors but not the other, it is said to be oscillating in a **normal mode**. Physically, this means that both types of motion occur with the same frequency, and they pass through their equilibrium positions at the same time.

If the system is set in motion in one of its normal (eigen) modes then it will continue to oscillate without any change in its translational or rotational amplitude (except that both amplitudes exponentially decay due to friction). The two normal modes have different frequencies, one higher and one lower than the resonant frequency. The difference in these two frequencies equals the beat frequency observed at resonance for energy transfer between translation and rotation.

If equations (2) and (4) are rewritten as $\mathbf{T}_{\mathbf{trans}} = \mathbf{2}\boldsymbol{\pi}\sqrt{\dfrac{\mathbf{M}_{\mathbf{eff}}}{\mathbf{k}}}$ and $\mathbf{T}_{\mathbf{rot}} = \mathbf{2}\boldsymbol{\pi}\sqrt{\dfrac{\mathbf{I}_{\mathbf{eff}}}{\mathbf{k}}}$ where $\mathbf{M}_{\mathbf{eff}}$ and $\mathbf{I}_{\mathbf{eff}}$ are given by equations (2) and (4) respectively then the normal modes may be constructed (see the Berg reference in Am. J. Phys. above) by setting

$$\mathbf{z}_0 = \pm\sqrt{\frac{\mathbf{I}_{\mathbf{eff}}}{\mathbf{M}_{\mathbf{eff}}}}\,\boldsymbol{\theta}_0 \qquad (5)$$

where $\mathbf{z}_0$ and $\boldsymbol{\theta}_0$ are the initial amplitudes for translation and rotation respectively.

The Normal Modes

a) Ensure that the cylindrical counterweight is in the correct position as determined in Part 1.

b) Determine $\mathbf{M_{eff}}$.

c) From your graph of $\mathbf{T^2_{rot}}$ versus $\mathbf{d^2}$ determine $\mathbf{I_0}$ and hence $\mathbf{I_{eff}}$.

d) Adjust the system for resonance and measure the beat frequency or the frequency for the transfer of energy between the two types of motion.

e) Using equation (5), determine the values of the amplitudes for a normal mode. Hint: choosing $\boldsymbol{\theta_0 = \pi}$ and then determining $\mathbf{z_0}$ is experimentally convenient. Set the system in motion with these values and observe whether or not it is oscillating in a normal mode.

f) Measure the frequencies of the normal modes. Do your results agree with the value for the beat frequency observed in (d)?

g) For the same $\boldsymbol{\theta_0}$, change $\mathbf{z_0}$ and investigate the effect of this change on the motion.

Checking Assumptions

Equation (2), [and by analogy, equation (3)] is an approximation which becomes better as **M** becomes large compared to $\mathbf{M_{sp}}$. Compare your measured value of $\mathbf{T_{trans}}$ with the value calculated using (2). You can increase **M** by sliding a slotted mass onto the top of the Wilberforce pendulum. Is (2) a good approximation?

Shear Modulus

The shear modulus **S** is related to the spring constant **k** by the formula $\mathbf{S = k\frac{4N\,R^3}{r^4}}$ where **r** is the radius of the wire, **R** is the radius of the helical coil and **N** is the number of turns in the coil. Calculate the shear modulus for the material of the spring and suggest what the material might be by comparing your value to those you can find in various references.

Young's Modulus

Young's modulus **Y** is related to the torsion constant $\boldsymbol{\kappa}$ for the spring as a whole by the formula $\mathbf{Y} = \boldsymbol{\kappa}\mathbf{\frac{8N\,R}{r^4}}$. Calculate Young's modulus for the material of the spring and suggest what the material might be by comparing your value to those you can find in various references.

(dh - 1974, jbv - 1990, jp - 2005)